Geometric Algebra for Mathematicians

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(this is an incomplete draft)

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1 Preface

A real vector space V allows the representation of 1-dimensional, weighted, oriented subspaces. A given vector $v \in V$ represents the subspace $\operatorname{span}(\{v\}) = \{\alpha v : \alpha \in \mathbb{R}\}$, called the *span* of v. Inside this subspace, vectors can be compared with respect to v; if $w = \alpha v$, for some $\alpha \in \mathbb{R}$, then its *magnitude* (relative to v) is $|\alpha|$ and its *orientation* (relative to v) is $\operatorname{sgn}(\alpha)$. To compare vectors between *different* subspaces requires a way to measure magnitudes. This is made possible by introducing a symmetric bilinear form in V, turning V into a symmetric bilinear space.

An *exterior algebra* on V extends V to a unital associative algebra $G(V)$, which allows the representation of subspaces of V as elements of $G(V)$. Each k-dimensional subspace of V can be represented in $G(V)$ as an exterior product of k vectors, and vice versa. Each element representing the same subspace in $G(V)$, the span of the element, can be compared to a baseline element to obtain its relative magnitude and relative orientation. A symmetric bilinear form in V induces a symmetric bilinear form in $G(V)$ (the scalar product), so that $G(V)$ also becomes a symmetric bilinear space. Given a symmetric bilinear form in $G(V)$, one may measure magnitudes of subspaces, and define the *contraction* in $G(V)$, which is the algebraic representation of the orthogonal complement of a subspace $A \subset V$ on a subspace $B \subset V$.

A *Clifford algebra* on V introduces a single product, the geometric product, in $G(V)$, which encompasses both the spanning properties required to form subspaces, and the metric properties required to measure magnitudes. A special property of this product over the more specific products is that it is associative. This brings in the structure of a unital associative algebra, which has the desirable properties of possessing a unique identity and unique inverses. Applying the geometric product in a versor product provides a structure-preserving way to perform reflections in V, which extends to orthogonal transformations in V by the Cartan-Dieudonné theorem.

1.1 Relation to other texts

This paper can be thought to complement the book *Geometric Algebra for Computer Science* [3], which we think is an excellent exposition of the state-of-the-art in geometric algebra, both in content and writing. A *mathematician*, however, desires for greater succinctness and detail; we aim to provide that in this paper. Ideally, one would alternate between this paper, for increased mathematical intuition, and that book, for increased geometric intuition.

The organization of this paper is inspired by the Schaum's Outlines series. In particular, we begin each section by giving the definitions without additional prose, and then follow up with a set of theorems, remarks, and examples concerning those definitions. In addition, we give many of the theorems and remarks a short summary; we hope this makes it more efficient to skip familiar parts later when recalling the subject matter.

In addition to [3], we merge together ideas from several other texts on geometric algebra, notably [4] and [2]. Here we briefly summarize the key points in which we have either adopted, or diverged from, an idea. A comparison between terms in different texts is given in Table 1.

Remark 1.1.1 (Coordinate-free proofs). We prove the results in Clifford algebra

Table 1: Comparison of terms between texts

Concept	This paper	[4]	[3]
Product of k vectors	Versor	Versor	-
Product of k invertible vectors	Invertible versor	Invertible versor	Versor
Simple element of $\mathcal{C}l(V)$	k-blade	simple k-vector	k-blade
Solutions to $v \wedge A = 0$	Span	?	Attitude
Measure of size in $\mathcal{C}l(V)$	Magnitude	Magnitude	Weight

without assuming a priviledged basis for the vectors. This reveals structure in proofs. The key theorem in this task is Theorem 4.5.14, which shows that vector-preserving homomorphisms preserve grade.

Remark 1.1.2 (General form for the bilinear form). We do not assume any particular form for the underlying symmetric bilinear form. This makes it explicit which theorems generalize to arbitrary symmetric bilinear forms, and which do not. In addition, this reveals structure in proofs.

Remark 1.1.3 (Contraction, not inner products). We adopt the definition of contraction from [3]. This definition is different from [4] and [2]. See Section 4.8 for details.

Remark 1.1.4 (Explicit notation for dual). We denote the dual of a k-blade A_k on an invertible l-blade B_l by ${A_k}^{\Vdash B_l}$. This notation is shown appropriate by Theorem 4.11.4, which says that

$$\operatorname{span}\left({A_k}^{\Vdash B_l}\right) = \operatorname{span}(A_k)^{\Vdash B_l}. \tag{1.1.1}$$

In addition, this makes the l-blade B_l explicit, which is important when taking duals in subspaces. We diverge from [3], which uses the notation A_k^*, and assumes the reader is aware of the l-blade B_l relative to which the dual is taken. The undual A_k^{-*} in [3] is given by ${A_k}^{\Vdash B_l^{-1}}$.

Remark 1.1.5 (Minor corrections). To translate results between this paper and [3], it is useful to be aware of some minor errors in their text:

- What they call a bilinear form is actually a symmetric bilinear form.
- What they call a degenerate bilinear form is actually an indefinite non-degenerate symmetric bilinear form, so of signature $(p, q, 0)$.
- What they call a metric space is actually a norm space.
- What they call a norm is correct, but only defined when the underlying symmetric bilinear form is positive-definite.

These errors are minor, and in no way affect the excellent readability of [3].

1.2 Acknowledgements

My interest on geometric algebra grew from my interest in geometric algorithms and computer graphics. Specifically, I was led to consider the question of whether there was some good way to obtain the intersection of two affine flats. After having queried this in the then active `comp.graphics.algorithms` USENET newsgroup, I was replied by a knowledgeable person named `Just'd'Faqs`, who suggested that I take a look at geometric algebra. This got me interested in geometric algebra, but unfortunately I did not have the time to study it back then. In 2010, I started my doctoral studies on geometric algebra (although I eventually changed the subject). It is this way that I was led to my doctoral supervisor, professor *Sirkka-Liisa Eriksson*, and to work as an exercise assistant on her course on geometric algebra. This paper got its start during the Christmas holiday 2011, as an attempt to clarify things to myself, before the start of the geometric algebra course in Spring 2012. I found the process such a good learning experience that I decided to continue on the writing on my free time. However, it wasn't until the end of 2012 that I would pick up on writing again, the motivation being the same geometric algebra course in Spring 2013. This paper, then, is the result of that process. I would like to thank `Just'd'Faqs` for guiding me to this direction, and Sirkka-Liisa Eriksson, for introducing me to the subject through her course. This paper wouldn't exist without the excellent books of Hestenes et al. [4], Dorst et al. [3], and Doran et al. [2]; these are the texts from which I have absorbed my knowledge.

Table 2: Latin alphabet (small)

Symbol	Meaning	Example
a, b, c, d	Vector in a vector space, 1-vector	$a \in V$
e	Exponential function	e^{B_2}
f, g, h	Function	$f(x)$, $g \circ h$
i, j	Index	$i \in I$
k, l, m	Cardinality of a set, grade of a k-vector	$\{a_1, \ldots, a_k\}$, A_k
n	Dimension of a vector space	
o	Avoided; resembles zero	
p, q, r	A distinguished point	$\lim_{x \to p} f(x) = y$
s, t	-	
u, v, w	Vector in a vector space	
x, y, z	Point in a topological space	
$\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$	Represented point in the conformal model	
$[\mathbf{p}]$	Point representative in the conformal model	$p \in [\mathbf{p}]$

Table 3: Latin alphabet (capitals)

Symbol	Meaning	Example
A, B, C	Multi-vector	$A = \sum_{k=0}^{n} A_k$
A_k, B_l, C_m	k-vector	
B	Basis of a vector space or of a topological space	$B_X \subset T_X$
C	Closed sets of a topological space	C_X
D, E	-	
F	Field	$\alpha \in F$
G, H	Group	
I, J	Index set	$\{a_i\}_{i \in I} \subset V$
K	-	
L	Linear functions between vector spaces	$L(U, V)$
M, N, O	Neighborhood, open set	$O_p \in T_X(p)$
P, Q	-	
R	Rotor	
S	Subset, subspace	$S \subset V$
T	Topology	T_X
U, V, W	Vector space	$v \in V$
X, Y, Z	Topological space	$x \in X$

Table 4: Greek alphabet (small)

Symbol	Meaning	Example
α, β, γ	Scalar, scalar tuple	$\alpha \in F$, $\beta \in F^n$
δ, ϵ	Small scalar, especially in limits	
ζ, η, θ, ι, κ	-	
λ	Scalar	
μ, ν, ξ	-	
o	Avoided; resembles zero	
π	Pi constant	
ρ	Scalar, radius	
σ	Permutation	$\sigma \in \sigma(S)$
τ, υ	-	
ϕ	Homomorphism	$\phi : V \to W$
χ	-	
ψ	Homomorphism	$\psi : V \to W$
ω	-	

Table 5: List of named structures

Symbol	Meaning	Example
$\mathbb{N}$	Natural numbers	$0, 1, 2$
$\mathbb{Z}$	Integers	$-2, -1, 0, 1, 2$
$\mathbb{R}$	Real numbers	
$\mathbf{GL}(V)$	General linear group of V	
$\mathbf{S}(V)$	Scaling linear group of V	
$\mathbf{O}(V)$	Orthogonal linear group of V	
$\mathcal{CO}(V)$	Conformal linear group of V	
$\mathbf{T}(V)$	Translation affine group of V	

Table 6: List of operators

Symbol	Meaning	Example
$\widehat{\ }$	Grade involution	$\widehat{A}$
$\widetilde{\ }$	Reversion	$\widetilde{A}$
$\overline{\ }$	Conjugation	$\overline{A}$
$\cdot$	Dot product, bilinear form	$x \cdot y$
$*$	Scalar product	$A * B$
$\wedge$	Exterior product	$A \wedge B$
$\rfloor$	Left contraction	$A \rfloor B$
$\lfloor$	Right contraction	$A \lfloor B$
$\times$	Commutator product	$A \times B$
$\rtimes$	Semi-direct product of groups	$\mathbf{T}(V) \rtimes \mathbf{O}(V)$
$\dot{+}$	Internal direct sum of vector spaces	$V \dot{+} W$
$\oplus$	External direct sum of vector spaces	$V \oplus W$
$\perp$	Orthogonal sum of bilinear spaces	$V \perp W$
$\otimes$	Tensor product of vector spaces	$V \otimes W$
$/$	Quotient vector space	V/W
$\setminus$	Set difference	$\mathbb{R} \setminus \{0\}$
${}^{\Vdash V}$	Orthogonal complement on subspace V	$S^{\Vdash V}$
${}^{\Vdash B}$	Dual on $B \in \mathcal{C}l(V)$	$A^{\Vdash B}$
$\lvert \cdot \rvert$	Cardinality of a set	$\lvert S \rvert$
$\mathcal{P}(\cdot)$	Powerset of a set	$\mathcal{P}(X)$
$\overline{\ }$	Closure of a set	$\overline{S}$
Δ_p	Differencing at point p	$\Delta_p(f)$
$\mathcal{D}_p$	Differentiation at point p	$\mathcal{D}_p(f)$
$\circ$	Composition of functions	$f \circ g$

2 Preliminaries

The aim of this paper is to study Clifford algebra, which is a unital associative algebra over a field. Since this algebra contains a symmetric bilinear form, the preliminary theory which enters their study consists not only of vector spaces, but also of bilinear spaces. Of the bilinear spaces those with finite dimension, symmetry, and non-degenerateness are priviledged over others. The reason for this is that it is only here where the orthogonal complement behaves in a geometrically meaningful manner, and where orthogonal functions can be represented as compositions of reflections. These properties are both essential for geometric algebra. In this section we develop these preliminary theories in full generality. This approach shows clearly which properties generalize, and which do not. We will assume that the reader is familiar on a basic level with groups and fields.

2.1 Basic definitions

A **norm** on a field F is a function $|\cdot| : F \to \mathbb{R}$ such that

- $|x| = 0 \Leftrightarrow x = 0$,
- $|x| \geq 0$,
- $|xy| = |x||y|$,
- $|x + y| \leq |x| + |y|$,

for all $x, y \in F$. A **normed field** F is a field together with a norm in F. The **characteristic of a ring** R, denoted by $\mathrm{char}(R)$, is the minimum number of times the identity element $1 \in R$ needs to be added to itself to obtain the zero element. If there is no such number, then $\mathrm{char}(R) = 0$. Let X be a set. The set of subsets of X is denoted by $\mathcal{P}(X)$. A **permutation** of X is a bijective function $\sigma : X \to X$. The set of permutations of X is denoted by

$$\sigma(X) = \{\sigma : X \to X : \sigma \text{ is bijective}\}. \tag{2.1.1}$$

A function is called an **involution** if it is its own inverse.

Example 2.1.1. The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are both normed fields, whose norms are given by the absolute value.

Remark 2.1.2 (Two needs to be invertible). The characteristic of a ring is of interest to us mainly because we want $1 + 1 \in R$ to be invertible (strictly speaking, the element 2 might not exist in R, unless interpreted as $1 + 1$). Many of the results in this paper fail if this is not the case.

Theorem 2.1.3 (Group equality by surjective homomorphisms). *Let G, G', and H be groups such that G is a sub-group of G'. Let $f' : G' \to H$ and $f : G \to H$ be surjective homomorphisms such that $f = f'|G$, and $f^{-1}(1) = f'^{-1}(1)$. Then $G = G'$.*

Proof. Let $g' \in G'$. Since f is surjective, there exists $g \in G$ such that $f(g) = f'(g')$. Since $f = f'|G$, it holds that $f(g) = f'(g)$. Therefore

$$\begin{aligned} & f'(g') = f'(g) \\ \Leftrightarrow & f'(g')f'(g)^{-1} = 1 \\ \Leftrightarrow & f'(g')f'(g^{-1}) = 1 \\ \Leftrightarrow & f'(g'g^{-1}) = 1 \\ \Leftrightarrow & g'g^{-1} \in f'^{-1}(1) = f^{-1}(1) \subset G. \end{aligned} \tag{2.1.2}$$

Therefore $g' = (g'g^{-1})g \in G$. ☐

2.2 Vector spaces

A **vector space** over the field F is a commutative group V together with a field of group endomorphisms $\{\phi_\alpha : V \to V\}_{\alpha \in F}$, such that $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta$, for all $\alpha, \beta \in F$, and ϕ_1 is the identity function. For brevity, we will identify $\alpha \in F$ with ϕ_α, and denote $\alpha v = \phi_\alpha(v)$, for all $v \in V$. The V is called **trivial**, if $V = \{0\}$. If I is any set, then we denote

$$\widehat{F^I} = \left\{\alpha \in F^I : |\{i \in I : \alpha(i) \neq 0\}| \in \mathbb{N}\right\} \tag{2.2.1}$$

Let $A = \{a_i\}_{i \in I} \subset V$. The **span** of A is the set of all finite linear combinations of A

$$\operatorname{span}(A) = \left\{\sum_{i \in I} \alpha_i a_i : \alpha \in \widehat{F^I}\right\}, \tag{2.2.2}$$

The A is said to *generate* V, if $\operatorname{span}(A) = V$, and to be **linearly independent**, if

$$\sum_{i \in I} \alpha_i a_i = 0 \Leftrightarrow \alpha = 0, \tag{2.2.3}$$

for all $\alpha \in \widehat{F^I}$. If A is not linearly independent, then it is **linearly dependent**. A **basis** of V is a maximal linearly independent subset of V. The **dimension** of V is the cardinality of any basis of V, denoted by $\dim(V)$. Let W be a vector space over F. A **subspace** of V is a subset $W \subset V$ such that $\operatorname{span}(W) = W$. Let $U, W \subset V$ be subspaces of V. The **sum** of U and W is the subspace

$$U + W = \{u + w : u \in U \text{ and } w \in W\} \subset V. \tag{2.2.4}$$

If $V = U + W$, and every vector $v \in V$ can be given as a *unique* sum $v = u + w$, where $u \in U$, and $w \in W$, then we will denote $V = U \dotplus W$, and say that V is the **internal direct sum** of U and W. The **external direct sum** $V \oplus W$ of vector spaces V and W is the external direct sum of groups V and W, together with group endomorphisms of the form

$$\alpha((v, w)) = (\alpha v, \alpha w), \tag{2.2.5}$$

for all $\alpha \in F$, $v \in V$, and $w \in W$. If $W \subset V$ is a subspace of V, then the **quotient space** V/W is the quotient group $\{v + W\}_{v \in V}$, together with group endomorphisms of the form

$$\alpha(v + W) = \alpha v + W, \tag{2.2.6}$$

for all $\alpha \in F$, and $v \in V$. A function $f : V \to W$ is called **linear** if

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \tag{2.2.7}$$

for all $\alpha, \beta \in F$, and $u, v \in V$. The **rank** of f is the dimension of the subspace $f(V)$, and the **nullity** of f is the dimension of the subspace $f^{-1}\{0\}$. The **dual space** V^* of V is the vector space of linear functions from V to F. The **free vector space** over F on set I is the set

$$\big\{f : I \to F : |f^{-1}(F \setminus \{0\})| \in \mathbb{N}\big\}, \tag{2.2.8}$$

together with point-wise addition and scalar-multiplication. The linear functions are the homomorphisms of vector spaces.

Example 2.2.1. If V is trivial, then $B = \emptyset \subset V$ is a basis of V; it is vacuously linearly independent, and also maximal, since $B \cup \{0\}$ is linearly dependent. The span of the empty set is $\operatorname{span}(\emptyset) = \{0\}$.

Remark 2.2.2. By definition, all bases of V have equal cardinality.

Remark 2.2.3. The quotient is the inverse of the direct sum; it subtracts a subspace. That is achieved by identification of vectors.

Remark 2.2.4 (Direct sum is the direct product). The direct sum and the direct product of groups and vector spaces are the same thing. The additive terminology is used when dealing with commutative groups, and the multiplicative terminology is used when dealing with non-commutative groups. Since a vector space is built on a commutative group, the additive terminology is adopted for vector spaces.

Theorem 2.2.5 (Every vector space has a basis). *Let V be a vector space, and $A \subset V$ be a linearly independent set. Then there exists a basis $B \subset V$ of V such that $A \subset B$.*

Proof. Let L be the set of those linearly independent subsets of V which contain A. Then L is partially ordered by the inclusion relation. Let $C \subset L$ be a linearly ordered subset of L, and $\hat{S} = \bigcup_{S \in C} S$. Since every $S \in C$ is contained in $\hat{S}$, every subset of $\hat{S}$ is linearly independent, and $A \subset \hat{S}$, $\hat{S} \in L$. Therefore $\hat{S}$ is an upper bound of C in L. By Zorn's lemma, there exists a maximal element $B \in L$. Suppose B is not maximal as a linearly independent set. Then there exists $v \in V$ such that $B \cup \{v\}$ is linearly independent. Since $A \subset B \cup \{v\}$, this contradicts the maximality of B. Therefore B is a basis. $\square$

Remark 2.2.6. In particular, Theorem 2.2.5 shows that every vector space has a basis, since the empty set is vacuously linearly independent.

Remark 2.2.7. For finite-dimensional vector spaces, Theorem 2.2.5 can also be shown without using Zorn's lemma. The claim that *every* vector space has a basis is equivalent to Zorn's lemma. Other equivalents include the axiom of choice, and that cardinal numbers can be linearly ordered.

Remark 2.2.8. Every free vector space has a basis. Therefore there exists vector spaces of arbitrary dimensions even without assuming Zorn's lemma.

Theorem 2.2.9 (Basis is linearly independent and generating). *Let V be a vector space over F, and $B \subset V$. Then B is a basis if and only if B is linearly independent and generates V.*

Proof. Let $B = \{b_i\}_{i \in I} \subset V$, and assume B is a basis. Then B is linearly independent by definition. Suppose B does not generate V; then there exists $v \in V \setminus \text{span}(B)$. Consider the equation

$$\alpha v + \sum_{i \in I} \beta_i b_i = 0, \tag{2.2.9}$$

where $\alpha \in F$, and $\beta \in \widehat{F^I}$. If $\alpha \neq 0$, then v can be solved in terms of B, which contradicts v not being in the span of B. Therefore $\alpha = 0$. Since B is linearly independent, $\beta = 0$. Therefore $\{v\} \cup B$ is linearly independent, which contradicts the maximality of B. Therefore B generates V. Assume B is linearly independent and generates V. Then any $v \in V$ can be solved in terms of B, and so $\{v\} \cup B$ is linearly dependent. Therefore B is maximal. $\square$

Theorem 2.2.10 (Linear independence is equivalent to unique coordinates). *Let V be a vector space over F, and $B = \{b_i\}_{i \in I} \subset V$. Then B is linearly independent if and only if for each $v \in$ span(B) there exists a unique $\alpha \in \widehat{F^I}$ such that $v = \sum_{i \in I} \alpha_i b_i$.*

Proof. Assume B is linearly independent. Since $v \in \text{span}(B)$, there exists $\alpha \in \widehat{F^I}$ such that

$$v = \sum_{i \in I} \alpha_i b_i. \tag{2.2.10}$$

Suppose there exists $\beta \in \widehat{F^I}$ such that

$$v = \sum_{i \in I} \beta_i b_i. \tag{2.2.11}$$

By subtraction, it then follows that

$$\sum_{i \in I} (\alpha_i - \beta_i) b_i = 0. \tag{2.2.12}$$

Since B is linearly independent, $\alpha = \beta$. Therefore α is unique. Assume that for each $v \in \text{span}(B)$ there exists a unique $\alpha \in \widehat{F^I}$ such that $v = \sum_{i \in I} \alpha_i b_i$, and consider the equation

$$\sum_{i \in I} \alpha_i b_i = 0. \tag{2.2.13}$$

Since $\alpha = 0$ is a solution to this equation, and that solution is unique, necessarily $\alpha = 0$. Therefore B is linearly independent. $\square$

Theorem 2.2.11 (Image and kernel are subspaces). *Let V and W be vector spaces over F, and $f : V \to W$ be linear. Then $f(V)$ is a subspace of W, and $f^{-1}\{0\}$ is a subspace of V.*

Proof. Let $x, y \in V$, and $\alpha, \beta \in F$. Then

$$\alpha f(x) + \beta f(y) = f(\alpha x + \beta y) \in f(V). \tag{2.2.14}$$

Therefore $f(V)$ is a subspace of W. If in addition $f(x) = 0$ and $f(y) = 0$, then

$$\begin{aligned} f(\alpha x + \beta y) &= \alpha f(x) + \beta f(y) \\ &= 0. \end{aligned} \tag{2.2.15}$$

Therefore $f^{-1}\{0\}$ is a subspace of V. $\square$

Theorem 2.2.12 (Rank-nullity theorem). *Let V and W be vector spaces, and $f : V \to W$ be linear. Then*

$$dim(V) = dim(f(V)) + dim\big(f^{-1}\{0\}\big). \tag{2.2.16}$$

Proof. Let $A = \{a_i\}_{i \in I} \subset W$ be a basis of $f(V)$, and $B = \{b_j\}_{j \in J} \subset V$ be a basis of $f^{-1}\{0\}$. Let $C = \{c_i\}_{i \in I} \subset V$ be such that $f(c_i) = a_i$, for all $i \in I$, and $x \in V$. Since A generates $f(V)$, there exists $\alpha \in \widehat{F^I}$, such that

$$f(x) = \sum_{i \in I} \alpha_i a_i. \tag{2.2.17}$$

Then

$$\begin{aligned} f\left(x - \sum_{i \in I} \alpha_i c_i\right) &= f(x) - \sum_{i \in I} \alpha_i f(c_i) \\ &= f(x) - \sum_{i \in I} \alpha_i a_i \\ &= 0. \end{aligned} \tag{2.2.18}$$

and therefore $x - \sum_{i \in I} \alpha_i c_i \in f^{-1}\{0\}$. Since B generates $f^{-1}\{0\}$, there exists $\beta \in \widehat{F^J}$, such that

$$x - \sum_{i \in I} \alpha_i c_i = \sum_{j \in J} \beta_j b_j. \tag{2.2.19}$$

Therefore

$$x = \sum_{i \in I} \alpha_i c_i + \sum_{j \in J} \beta_j b_j, \tag{2.2.20}$$

and $B \cup C$ generates V. Consider the equation

$$\sum_{i \in I} \alpha_i c_i + \sum_{j \in J} \beta_j b_j = 0. \tag{2.2.21}$$

Then

$$\begin{aligned} f\left(\sum_{i \in I} \alpha_i c_i + \sum_{j \in J} \beta_j b_j\right) &= \sum_{i \in I} \alpha_i f(c_i) \\ &= \sum_{i \in I} \alpha_i a_i \\ &= 0. \end{aligned} \tag{2.2.22}$$

Since A is linearly independent, $\alpha = 0$. Since B is linearly independent, $\beta = 0$. Therefore $C \cup B$ is linearly independent, and a basis of V. Then

$$\begin{aligned} \dim(V) &= |C \cup B| \\ &= |C| + |B| \\ &= |A| + |B| \\ &= \dim(f(V)) + \dim\bigl(f^{-1}\{0\}\bigr). \end{aligned} \tag{2.2.23}$$

□

Theorem 2.2.13 (Subset of a linearly independent set is linearly independent). *Let V be a vector space over F, $A, B \subset V$ be such that B is linearly independent, and $A \subset B$. Then A is linearly independent.*

Proof. Let $B = \{b_i\}_{i \in I} \subset V$, and $A = \{b_j\}_{j \in J} \subset B$. Consider the equation

$$\sum_{j \in J} \alpha_j b_j = 0, \tag{2.2.24}$$

where $\alpha \in \widehat{F^J}$. This equation is equivalent to

$$\sum_{j \in J} \alpha_j b_j + \sum_{i \in I \setminus J} 0 b_i = 0. \tag{2.2.25}$$

Since B is linearly independent, $\alpha = 0$. Therefore A is linearly independent. □

Theorem 2.2.14 (Superset of a generating set is generating). *Let V be a vector space over F, $A, B \subset V$ be such that B generates V, and $B \subset A$. Then A generates V.*

Proof. If $A = B$, then the result holds trivially. Assume $A \neq B$. Holding the coefficients of $A \setminus B$ zero, we see that $\operatorname{span}(B) \subset \operatorname{span}(A)$. Since $\operatorname{span}(B) = V$, $\operatorname{span}(A) = V$. □

Theorem 2.2.15 (Superset of a basis is not linearly independent). *Let V be a vector space over F, $A, B \subset V$ be such that B is a basis of V, and $A \supsetneq B$. Then A is linearly dependent.*

Proof. Suppose A is linearly independent. Then B is not a maximal linearly independent set; a contradiction. Therefore A is linearly dependent. □

Theorem 2.2.16 (Subset of a basis is not generating). *Let V be a vector space over F, $A, B \subset V$ be such that B is a basis of V, and $A \subsetneq B$. Then A does not generate V.*

Proof. Theorem 2.2.13 shows that A is linearly independent. Suppose A generates V. Then Theorem 2.2.15 shows that B is linearly dependent; a contradiction. Therefore A does not generate V. □

Theorem 2.2.17 (Vector-space-isomorphic is equivalent to equal dimensions). *Let V and W be vector spaces over F. Then V and W are isomorphic if and only if $dim(V) = dim(W)$.*

Proof. Assume V and W are isomorphic. Let $f : V \to W$ be a bijective linear function, and $B = \{b_i\}_{i \in I} \subset V$ be a basis of V. Let $\alpha \in \widehat{F^I}$, and consider the equation

$$\begin{aligned} \sum_{i \in I} \alpha_i f(b_i) &= f\left(\sum_{i \in I} \alpha_i b_i\right) \\ &= 0. \end{aligned} \tag{2.2.26}$$

Since f is bijective, this holds if and only if

$$\sum_{i \in I} \alpha_i b_i = 0. \tag{2.2.27}$$

Since B is linearly independent, $\alpha = 0$. Therefore $f(B)$ is linearly independent. Since f is surjective, $f(B)$ generates V. Therefore $f(B)$ is a basis of W, and $\dim(V) = \dim(W)$. Assume $\dim(V) = \dim(W)$. Let $B = \{b_i\}_{i \in I} V$ be a basis of V, and $C = \{c_i\}_{i \in I} \subset W$ be a basis of W. Let $f : V \to W$ be a linear function such that

$$f(b_i) = c_i, \tag{2.2.28}$$

for all $i \in I$. Consider the equation

$$\begin{aligned} f\left(\sum_{i \in I} \alpha_i b_i\right) &= \sum_{i \in I} \alpha_i f(b_i) \\ &= \sum_{i \in I} \alpha_i c_i \\ &= 0, \end{aligned} \tag{2.2.29}$$

where $\alpha \in \widehat{F^I}$. Since C is linearly independent, $\alpha = 0$; therefore f is injective. Since C generates V, the second line shows that f is surjective. Therefore f is an isomorphism. □

Theorem 2.2.18 (Dimension of quotient space). *Let V be a vector space over F, and $W \subset V$ be a subspace of V. Then $\dim(V/W) = \dim(V) - \dim(W)$.*

Proof. Let $f : V \to W$ be a surjective linear function, and $V = W \dotplus U$. Then $W = f(V)$, $f^{-1}\{0\} = U$, and $\dim(V) = \dim(W) + \dim(U)$ by Theorem 2.2.12. Since U is isomorphic to V/W by the canonical map, $\dim(U) = \dim(V/W)$ by Theorem 2.2.17. □

Theorem 2.2.19 (Vector space is isomorphic to finitely-non-zero field-sequences). *Let V be a vector space over F, and I be a set such that $|I| = \dim(V)$. Then $V \cong \widehat{F^I}$.*

Proof. Let $B = \{b_i\}_{i \in I} \subset V$ be a basis of V, and $v \in V$. Since B is a basis of V, there exists a unique $\alpha \in \widehat{F^I}$, called the coordinates of v in B, such that

$$v = \sum_{i \in I} \alpha_i b_i. \tag{2.2.30}$$

Let $f : V \to \widehat{F^I}$ be the function which sends each $v \in V$ to its coordinates in B. Then by Theorem 2.2.10 f is injective. For each $\alpha \in \widehat{F^I}$ one can compute a corresponding v by the above sum. Therefore f is surjective. Since f is also linear, $\widehat{F^I}$ is isomorphic to V. □

Theorem 2.2.20 (Dual space is isomorphic to field-sequences). *Let V be a vector space over F, and I be a set such that $|I| = dim(V)$. Then $V^* \cong F^I$.*

Proof. Let $\{b_i\}_{i \in I} \subset V$ be a basis of V, and $h_f : I \to F$ be such that

$$h_f(i) = f(b_i), \tag{2.2.31}$$

for all $f \in V^*$. Let $\phi : V^* \to F^I$ be such that

$$\phi(f) = h_f. \tag{2.2.32}$$

Then if $g \in V^*$, and $\alpha, \beta \in F$,

$$\begin{aligned} \phi(\alpha f + \beta g)(i) &= h_{\alpha f + \beta g}(i) \\ &= (\alpha f + \beta g)(b_i) \\ &= \alpha f(b_i) + \beta g(b_i) \\ &= \alpha h_f(i) + \beta h_g(i) \\ &= (\alpha \phi(f) + \beta \phi(g))(i), \end{aligned} \tag{2.2.33}$$

and therefore ϕ is linear. Suppose $\phi(f) = \phi(g)$. This is equivalent to $f(b_i) = g(b_i)$, for all $i \in I$. Let $v \in V$. Since B is a basis, there exists $\alpha \in \widehat{F^I}$ such that

$$v = \sum_{i \in I} \alpha_i v_i. \tag{2.2.34}$$

Then

$$\begin{aligned} f(v) &= f\left(\sum_{i \in I} \alpha_i v_i\right) \\ &= \sum_{i \in I} \alpha_i f(v_i) \\ &= \sum_{i \in I} \alpha_i g(v_i) \\ &= g\left(\sum_{i \in I} \alpha_i v_i\right) \\ &= g(v). \end{aligned} \tag{2.2.35}$$

Therefore ϕ is injective. On the other hand, if $\beta \in F^I$, then

$$f(v) = \sum_{i \in I} \alpha_i \beta_i \tag{2.2.36}$$

defines a function $f \in V^*$. Therefore f is surjective, and ϕ is an isomorphism. □

Theorem 2.2.21 (Vector space and its dual space are isomorphic when finite-dimensional). *Let V be a vector space over F. Then V is finite-dimensional if and only if $V \cong V^*$.*

Proof. Let I be a set such that $|I| = \dim(V)$. Then $V \cong \widehat{F^I}$ by Theorem 2.2.19, and $V^* \cong F^I$ by Theorem 2.2.20. Therefore it suffices to show that $\widehat{F^I} \cong F^I$ if and only if I is finite. Assume I is finite. Then $\widehat{F^I} = F^I$, and therefore $V \cong V^*$. Assume I is infinite. Suppose F is finite. Then

$$|F| < \dim\left(\widehat{F^I}\right) = |I| \leq \dim\left(F^I\right). \tag{2.2.37}$$

Suppose F is infinite. Let $B = \{b_i\}_{i \in I} \subset V$ be a basis of V, and $g : \mathbb{N} \to I$ be an injective function. Let $C = \{f_\alpha \in F^I\}_{\alpha \in F \setminus \{0\}}$ be such that

$$f_\alpha(i) = \begin{cases} \alpha^j, & \text{if } j \in g^{-1}\{i\} \\ 0, & \text{otherwise }. \end{cases} \tag{2.2.38}$$

Let $\alpha \in F^k$, such that $\alpha_i \neq \alpha_j$, for $i \neq j$, and consider the equation

$$\sum_{i=1}^{k} \beta_i f_{\alpha_i} = 0. \tag{2.2.39}$$

By construction, this implies that for every $m \in \mathbb{N}$ we have

$$\sum_{i=1}^{k} \beta_i \alpha_i^m = 0. \tag{2.2.40}$$

Considering the first k such equations, we get a matrix equation of the form

$$\begin{bmatrix} \alpha_1^1 & \dots & \alpha_k^1 \\ \vdots & \ddots & \vdots \\ \alpha_1^k & \dots & \alpha_k^k \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.2.41}$$

The coefficient matrix is a Vandermonde matrix, whose determinant is non-zero, since $\alpha_i \neq \alpha_j$, when $i \neq j$. Therefore $\beta = 0$ is the only solution, and $\{f_{\alpha_1}, \dots, f_{\alpha_k}\} \subset V^*$ is linearly independent. It follows that C is linearly independent, and $|F| = |C| \leq \dim\left(F^I\right)$. Therefore, whether $|F|$ is finite or infinite,

$$\begin{aligned} |F^I| &= \dim\left(F^I\right)|F| \\ &= \max(\dim\left(F^I\right), |F|) \\ &= \dim\left(F^I\right). \end{aligned} \tag{2.2.42}$$

It follows that

$$\begin{aligned} \dim\left(\widehat{F^I}\right) &= |I| \\ &< 2^{|I|} \\ &\leq |F|^{|I|} \\ &= \dim\left(F^I\right). \end{aligned} \tag{2.2.43}$$

Therefore $\widehat{F^I}$ is not isomorphic to F^I by Theorem 2.2.17. $\square$

Theorem 2.2.22 (Solvability is linear dependence). *Let V be a vector space over F, and $S \subset V$. Then S is linearly dependent if and only if there exists $v \in S$ such that $v \in span(S \setminus \{v\})$.*

Proof. Let $S = \{v_i\}_{i \in I} \subset V$. If S is empty, then the result holds; assume S is not empty. Assume S is linearly dependent. Then there exists $\alpha \in \widehat{F^I} \setminus \{0\}$ such that

$$\sum_{i \in I} \alpha_i v_i = 0. \tag{2.2.44}$$

Since some $\alpha_j \neq 0$, we may solve

$$v_j = -\frac{1}{\alpha_j} \sum_{i \in I \setminus \{j\}} \alpha_i v_i. \tag{2.2.45}$$

Therefore $v_j \in \text{span}(S \setminus \{v_j\})$. Assume $v_j \in \text{span}(S \setminus \{v_j\})$. Then there exists $\alpha \in \widehat{F^{I \setminus \{j\}}}$ such that

$$v_j = \sum_{i \in I \setminus \{j\}} \alpha_i v_i. \tag{2.2.46}$$

This can be rewritten as

$$v_j - \sum_{i \in I \setminus \{j\}} \alpha_i v_i = 0. \tag{2.2.47}$$

Since the coefficient of v_j is non-zero, S is linearly dependent. □

Theorem 2.2.23 (Introducing linear dependence is being in span). *Let V be a vector space over F, $B \subset V$ be linearly independent, and $v \in V$. Then*

$$v \in span(B) \Leftrightarrow \{v\} \cup B \text{ is linearly dependent.} \tag{2.2.48}$$

Proof. Let $B = \{b_i\}_{i \in I} \subset V$. Assume $v \in \text{span}(B)$. Then Theorem 2.2.22 shows that $\{v\} \cup B$ is linearly dependent. Assume $\{v\} \cup B$ is linearly dependent, and consider the equation

$$\alpha v + \sum_{i \in I} \beta_i b_i = 0, \tag{2.2.49}$$

where $\alpha \in F$, and $\beta \in \widehat{F^I}$. If $\alpha = 0$, then the linear independence of B implies $\beta = 0$. Therefore $\{v\} \cup B$ is linearly independent; a contradiction. Therefore $\alpha \neq 0$. But then we may solve

$$v = \sum_{i \in I} \left(-\frac{\beta_i}{\alpha} \right) b_i. \tag{2.2.50}$$

Therefore $v \in \text{span}(B)$. □

Remark 2.2.24. The vectors in a free vector space on X represent formal finite linear combinations of the elements of X.

2.3 Bilinear spaces

Let V be a vector space over the field F. A **bilinear form** in V is a function $\cdot : V^2 \to F$ which is linear in both arguments. A **bilinear space** is a vector space V over the field F together with a bilinear form in V. A bilinear form in V, and the V itself, is called

- **symmetric**, if $\forall x, y \in V : x \cdot y = y \cdot x$,
- **skew-symmetric**, if $\forall x, y \in V : x \cdot y = -y \cdot x$,
- **alternating**, if $\forall x \in V : x \cdot x = 0$, and
- **reflexive**, if $\forall x, y \in V : x \cdot y = 0 \Leftrightarrow y \cdot x = 0$.

The **left-radical** of a bilinear form in V, and of V, is the subspace

$$\mathrm{rad}_L(V) = \{u \in V : \forall v \in V : v \cdot u = 0\}, \tag{2.3.1}$$

and the **right-radical** of a bilinear form in V, and of V, is the subspace

$$\mathrm{rad}_R(V) = \{u \in V : \forall v \in V : u \cdot v = 0\}. \tag{2.3.2}$$

If the left-radical and the right-radical are equal, we may simply speak of the **radical** of a bilinear form in V, and of V, and denote it by $\mathrm{rad}(V)$. A bilinear form in V, and V itself, is called **left-non-degenerate** on $S \subset V$ if

$$\mathrm{rad}_L(V) \cap S = \{0\}, \tag{2.3.3}$$

right-non-degenerate on S if

$$S \cap \mathrm{rad}_R(V) = \{0\}, \tag{2.3.4}$$

and **non-degenerate** on S if it is both. A subset $S \subset V$ is called **null**, if $v \cdot v = 0$ for all $v \in S$, and **orthogonal** if

$$u \cdot v = 0 = v \cdot u, \tag{2.3.5}$$

for all $u, v \in S$ such that $u \neq v$. Let $U, W \subset V$ be subspaces of V. The **orthogonal** of U on W is defined by

$$U^{\Vdash W} = \{w \in W : \forall u \in U : u \cdot w = 0 = w \cdot u\}. \tag{2.3.6}$$

The **orthogonal sum** $V \perp W$ of two bilinear spaces V and W is the direct sum of V and W as vector spaces, together with the bilinear form $(V \perp W)^2 \to F$ such that

$$(v_1 + w_1) \cdot (v_2 + w_2) = v_1 \cdot v_2 + w_1 \cdot w_2, \tag{2.3.7}$$

where $v_1, v_2 \in V$, and $w_1, w_2 \in W$. Let W be a bilinear space over F. A linear function $f : V \to W$ is called **orthogonal function** if

$$f(u) \cdot f(v) = u \cdot v, \tag{2.3.8}$$

for all $u, v \in V$. The orthogonal functions are the homomorphisms of bilinear spaces.

Remark 2.3.1. We overload the notation $\cdot$ to mean different bilinear forms in different bilinear spaces. The used arguments reveal which bilinear form is in question.

Theorem 2.3.2 (Radical decomposition). *Let V be a bilinear space over F. Then there exists a left-non-degenerate subspace $W \subset V$ such that $V \cong W \perp rad_L(V)$.*

Proof. Let $B \subset \text{rad}(V)$ be a basis of $\text{rad}(V)$, and $C \subset V$ be a basis of V such that $B \subset C$. Let $W = \text{span}(C \setminus B)$. Then $V \cong W \perp \text{rad}(V)$. Let $w \in \text{rad}(W)$, $v \in V$, and $U = \text{rad}(V)$. Then

$$\begin{aligned} w \cdot v &= w \cdot v_W + w \cdot v_U \\ &= w \cdot v_W \\ &= 0, \end{aligned} \tag{2.3.9}$$

since $w \in \text{rad}(W)$. Therefore $w \in \text{rad}(V)$. Since also $w \in W$, $w = 0$. Therefore W is non-degenerate. □

Theorem 2.3.3 (An orthogonal non-null set is linearly independent). *Let V be a reflexive bilinear space over F, and $B \subset V$ be an orthogonal set of non-null vectors. Then B is linearly independent.*

Proof. Let $B = \{b_i\}_{i \in I}$, and $\alpha \in \widehat{F^I}$. Then for $j \in I$,

$$\begin{aligned} &\sum_{i \in I} \alpha_i b_i = 0 \\ \Rightarrow &\left(\sum_{i \in I} \alpha_i b_i \right) \cdot b_j = 0 \\ \Rightarrow &\sum_{i \in I} \alpha_i (b_i \cdot b_j) = 0 \\ \Rightarrow &\alpha_j (b_j \cdot b_j) = 0 \\ \Rightarrow &\alpha_j = 0. \end{aligned} \tag{2.3.10}$$

Therefore B is linearly independent. □

Example 2.3.4 (Orthogonal linearly dependent set). Let $\cdot$ be a reflexive bilinear form in V, $\text{char}(F) \neq 2$, and $B = \{b, b + b\} \subset V$ such that b is null, and $b \neq 0$. Then B is an orthogonal set, but not linearly independent. Therefore, if $B = \{b_1, \dots, b_n\} \subset V$ is an orthogonal set of n vectors, some of which are null, then B may or may not be a basis for V.

Remark 2.3.5 (Linear independence does not depend on the bilinear form). The definition of linear independence does not depend on the bilinear form. Therefore the converse of Theorem 2.3.3 does not hold; a linearly independent set may contain null vectors (but never 0). Actually, as Theorem 2.4.6 shows, if the bilinear form is degenerate, then the radical needs to be spanned by null vectors. But even if the bilinear form is non-degenerate, but indefinite, null-vectors may still exist and be part of a linearly independent set.

Theorem 2.3.6 (Non-degenerateness by subspaces). *Let V be a reflexive bilinear space, and let $U, W \subset V$ be subspaces of V, such that $V \cong U \perp W$. Then V is non-degenerate if and only if U and W are non-degenerate.*

Proof. Assume U and W are non-degenerate. Let $v \in \text{rad}(V)$. Then $v = v_U + v_W$, for some $v_U \in U$, and $v_W \in W$. Let $u \in U$. Then

$$\begin{aligned} u \cdot v &= u \cdot v_U + u \cdot v_W \\ &= u \cdot v_U \\ &= 0, \end{aligned} \tag{2.3.11}$$

since $v \in \text{rad}(V)$. Therefore $v_U \in \text{rad}(U)$. Since U is non-degenerate, $v_U = 0$. By the same argument for W, $v_W = 0$. Therefore $v = 0$, and V is non-degenerate. Assume V is non-degenerate. Let $u \in \text{rad}(U)$, and $v \in V$. Then

$$\begin{aligned} u \cdot v &= u \cdot v_U + u \cdot v_W \\ &= u \cdot v_U \\ &= 0, \end{aligned} \tag{2.3.12}$$

since $u \in \text{rad}(U)$. Therefore $u \in \text{rad}(V)$. Since V is non-degenerate, $u = 0$, and U is non-degenerate. By the same argument for W, W is non-degenerate. $\square$

Theorem 2.3.7 (Polarization identity). *If the bilinear form $\cdot$ is symmetric, and char$(F) \neq 2$, then*

$$x \cdot y = \frac{1}{4}[(x+y)\cdot(x+y) - (x-y)\cdot(x-y)]. \tag{2.3.13}$$

Proof. By bilinearity,

$$\begin{aligned} (x+y)\cdot(x+y) &= x \cdot x + x \cdot y + y \cdot x + y \cdot y, \\ (x-y)\cdot(x-y) &= x \cdot x - x \cdot y - y \cdot x + y \cdot y. \end{aligned} \tag{2.3.14}$$

Subtracting the second row from the first row gives

$$(x+y)\cdot(x+y) - (x-y)\cdot(x-y) = 2(x \cdot y) + 2(y \cdot x). \tag{2.3.15}$$

Since the bilinear form is symmetric, and 2 is invertible, the result holds. $\square$

Remark 2.3.8 (Quadratic forms correspond mostly to symmetric bilinear forms). The polarization identity can also be written in other equivalent forms, but this one has perhaps the most symmetric form. In particular, this theorem shows that, when $\text{char}(F) \neq 2$, the quadratic forms and the symmetric bilinear forms represent the same concept. We choose to use the symmetric bilinear forms, because their bilinearity makes them more comfortable to use in derivations.

Theorem 2.3.9 (All null is equivalent to trivial when non-degenerate). *Let V be a non-degenerate symmetric bilinear space over F, with char$(F) \neq 2$. Then every vector in V is null if and only if V is trivial.*

Proof. Assume every vector in V is null. Then Theorem 2.3.7 shows that $x \cdot y = 0$, for all $x, y \in V$. Therefore $\mathrm{rad}(V) = V$. Since V is non-degenerate, V is trivial. Assume V is trivial. Since $0 \in V$ is null, every vector in V is null. □

Theorem 2.3.10 (Alternating-symmetric decomposition for bilinear forms). *Let V be a bilinear space over F, with char$(F) \neq 2$. Then a bilinear form in V is a sum of an alternating bilinear form and a symmetric bilinear form.*

Proof. Let

$$\oplus : V^2 \to F : x \oplus y = \frac{1}{2}[(x \cdot y) + (y \cdot x)], \text{ and} \tag{2.3.16}$$

$$\ominus : V^2 \to F : x \ominus y = \frac{1}{2}[(x \cdot y) - (y \cdot x)]. \tag{2.3.17}$$

Then $x \cdot y = (x \oplus y) + (x \ominus y)$, and both $\oplus$ and $\ominus$ are bilinear. Now $\oplus$ is symmetric and $\ominus$ is skew-symmetric. Since $\mathrm{char}(F) \neq 2$, skew-symmetry is equivalent to alternation. □

Theorem 2.3.11 (Reflexive is equivalent to symmetric or alternating). *A bilinear form is reflexive if and only if it is symmetric or alternating.*

Proof. The following proof is from the book *Characters and groups* by Larry C. Grove. Reflexivity is equivalent to

$$\forall u, v, w \in V : (u \cdot v)(w \cdot u) = (v \cdot u)(u \cdot w). \tag{2.3.18}$$

Taking $u = v$, it follows that

$$\forall v, w \in V : (v \cdot v)[w \cdot v - v \cdot w] = 0. \tag{2.3.19}$$

We would like to to show that this implies alternation or symmetry:

$$[\forall v \in V : v \cdot v = 0] \text{ or } [\forall v, w \in V : w \cdot v - v \cdot w = 0]. \tag{2.3.20}$$

Assume this does not hold. Then

$$[\exists y \in V : y \cdot y \neq 0] \text{ and } [\exists x, z \in V : x \cdot z - z \cdot x \neq 0]. \tag{2.3.21}$$

The equation 2.3.19 holds in particular for combinations of x, y and z:

$$(x \cdot x)[z \cdot x - x \cdot z] = 0, \tag{2.3.22}$$

$$(z \cdot z)[x \cdot z - z \cdot x] = 0, \tag{2.3.23}$$

$$(y \cdot y)[x \cdot y - y \cdot x] = 0, \tag{2.3.24}$$

$$(y \cdot y)[z \cdot y - y \cdot z] = 0. \tag{2.3.25}$$

These equations imply that

$$x \cdot x = 0, \tag{2.3.26}$$

$$z \cdot z = 0, \tag{2.3.27}$$

$$x \cdot y = y \cdot x, \tag{2.3.28}$$

$$z \cdot y = y \cdot z. \tag{2.3.29}$$

Substitute $u = x$, $v = y$, and $w = z$ in equation 2.3.18. Then

$$\begin{aligned}
(x \cdot y)(z \cdot x) &= (y \cdot x)(x \cdot z) && (2.3.30)\\
&\Rightarrow && (2.3.31)\\
(x \cdot y)[z \cdot x - x \cdot z] &= 0 && (2.3.32)\\
&\Rightarrow && (2.3.33)\\
x \cdot y &= 0, && (2.3.34)
\end{aligned}$$

since $x \cdot y = y \cdot x$, and $z \cdot x - x \cdot z \neq 0$. Substitute $u = z$, $v = y$, and $w = x$ in equation 2.3.18. Then

$$\begin{aligned}
(z \cdot y)(x \cdot z) &= (y \cdot z)(z \cdot x) && (2.3.35)\\
&\Rightarrow && (2.3.36)\\
(z \cdot y)[x \cdot z - z \cdot x] &= 0 && (2.3.37)\\
&\Rightarrow && (2.3.38)\\
z \cdot y &= 0, && (2.3.39)
\end{aligned}$$

since $z \cdot y = y \cdot z$, and $z \cdot x - x \cdot z \neq 0$. Summarizing our results thus far, we have found out that for the x, y, and z of equation 2.3.21, it holds that

$$\begin{aligned}
x \cdot x &= 0, && (2.3.40)\\
y \cdot y &\neq 0, && (2.3.41)\\
z \cdot z &= 0, && (2.3.42)\\
x \cdot y &= y \cdot x = 0, && (2.3.43)\\
y \cdot z &= z \cdot y = 0, && (2.3.44)\\
x \cdot z &\neq z \cdot x. && (2.3.45)
\end{aligned}$$

Now

$$\begin{aligned}
x \cdot (y + z) &= x \cdot y + x \cdot z = x \cdot z, && (2.3.46)\\
(y + z) \cdot x &= y \cdot x + z \cdot x = z \cdot x. && (2.3.47)
\end{aligned}$$

By equation 2.3.21,

$$x \cdot (y + z) - (y + z) \cdot x \neq 0. \qquad (2.3.48)$$

Choose $v = y + z$ and $w = x$ in equation 2.3.19. Then

$$\begin{aligned}
((y + z) \cdot (y + z))[x \cdot (y + z) - (y + z) \cdot x] &= 0 && (2.3.49)\\
&\Rightarrow && (2.3.50)\\
(y + z) \cdot (y + z) &= 0. && (2.3.51)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(y + z) \cdot (y + z) &= y \cdot y + y \cdot z + z \cdot y + z \cdot z && (2.3.52)\\
&= y \cdot y \neq 0, && (2.3.53)
\end{aligned}$$

a contradiction. Thus equation 2.3.20 holds. Therefore $\cdot$ is symmetric or alternating. $\square$

Remark 2.3.12. Reflexivity seems like a natural property to require from a bilinear form. Therefore, if we are to choose a reflexive bilinear form, then by Theorem 2.3.11 the choice is between a symmetric bilinear form or an alternating bilinear form. Symmetry and alternation are a recurring theme in geometry. In general, symmetry encodes lengths and angles, while alternation encodes volume and linear independence.

Theorem 2.3.13 (Orthogonal complement is a subspace). *Let V be a reflexive bilinear space over F, and $S \subset V$ be a subspace of V. Then $S^{\Vdash V}$ is a subspace of V.*

Proof. Let $\alpha, \beta \in F$, $x, y \in S^{\Vdash V}$, and $z \in S$. Then

$$\begin{aligned} z \cdot (\alpha x + \beta y) &= \alpha(z \cdot x) + \beta(z \cdot y) \\ &= 0. \end{aligned} \tag{2.3.54}$$

Therefore $S^{\Vdash V}$ is a subspace of V. □

Theorem 2.3.14 (Double orthogonal complement is monotone). *Let V be a reflexive bilinear space over F, and $S \subset V$ be a subspace of V. Then*

$$S \subset S^{\Vdash V \Vdash V}. \tag{2.3.55}$$

Proof. Now

$$\begin{aligned} & x \in S^{\Vdash V \Vdash V} \\ \Leftrightarrow & \forall y \in S^{\Vdash V} : x \cdot y = 0 \\ \Leftrightarrow & \forall y \in V : \big(y \in S^{\Vdash V} \Rightarrow x \cdot y = 0\big) \\ \Leftrightarrow & \forall y \in V : ((\forall z \in S : z \cdot y = 0) \Rightarrow x \cdot y = 0) \\ \Leftarrow & x \in S. \end{aligned} \tag{2.3.56}$$

□

Example 2.3.15. In general, equality does not hold in Theorem 2.3.14. In particular, if $V = \operatorname{rad}(V) \dotplus W$, for some non-degenerate subspace $W \subset V$, and $\operatorname{rad}(V)$ is not trivial, then $W^{\Vdash V} = \operatorname{rad}(V)$, and $W^{\Vdash V \Vdash V} = V$.

Remark 2.3.16. For completeness, we could define the orthogonal quotient as follows. Let $W \subset V$ be a subspace of the bilinear space V, such that $V \cong W \perp W^{\Vdash V}$. The **orthogonal quotient** V/W is the quotient space of V and W as vector spaces, together with the bilinear form $(V/W)^2 \to F$ such that

$$(u_1 + W) \cdot (u_2 + W) = u_1 \cdot u_2, \tag{2.3.57}$$

where and $u_1, u_2 \in W^{\Vdash V}$. Since then $V/W \cong W^{\Vdash V}$, the notion is not so useful; we can use $W^{\Vdash V}$ directly. Alternatively, the orthogonal complement can be thought of as the orthogonal quotient.

2.4 Finite-dimensional bilinear spaces

Finite-dimensional bilinear spaces are of special interest, because for them we can prove intuitive theorems for the behaviour of the orthogonal complement. The source of these theorems relies on the isomorphy of the vector space with its dual, given in the Theorem 2.2.20.

Theorem 2.4.1 (Left-non-degenerate is right-non-degenerate when finite-dimensional). *Let V be a finite-dimensional bilinear space over F. Then V is left-non-degenerate if and only if it is right-non-degenerate.*

Proof. By the characteristic property of tensor products, the vector space B of bilinear forms in V is isomorphic to $(V \otimes V)^*$. Let $f : B \to (V \otimes V)^*$ be an isomorphism of vector spaces. Then

$$\begin{aligned}
&\exists v \in V \setminus \{0\} : v \in \mathrm{rad}_L(V) \\
\Leftrightarrow &\exists v \in V \setminus \{0\} : \forall u \in V : u \cdot v = 0 \\
\Leftrightarrow &\exists v \in V \setminus \{0\} : \forall u \in V : f(\cdot)(u \otimes v) = 0 \\
\Leftrightarrow &f(\cdot) \text{ is not invertible.}
\end{aligned} \tag{2.4.1}$$

Similarly one shows $\mathrm{rad}_R(V)$ non-trivial if and only if $f(\cdot)$ is not invertible. Therefore $\mathrm{rad}_L(V)$ is non-trivial if and only if $\mathrm{rad}_R(V)$ is non-trivial. TODO: not sure of the second-to-last line. ☐

Remark 2.4.2. Theorem 2.4.1 shows that in finite-dimensional bilinear spaces the concepts of non-degenerate, left-non-degenerate, and right-non-degenerate coincide, even if the bilinear form isn't reflexive.

Theorem 2.4.3 (Riesz representation theorem for bilinear spaces). *Let V be a left-non-degenerate bilinear space over F. Then V is finite-dimensional if and only if*

$$V^* = \{\phi_v : V \to F : \phi_v(x) = x \cdot v\}_{v \in V}. \tag{2.4.2}$$

Proof. Let $f : V \to V^*$ be such that

$$f(v) = \phi_v. \tag{2.4.3}$$

Then f is linear, since

$$\begin{aligned}
f(\alpha x + \beta y)(z) &= \phi_{\alpha x + \beta y}(z) \\
&= z \cdot (\alpha x + \beta y) \\
&= \alpha(z \cdot x) + \beta(z \cdot y) \\
&= \alpha\phi_x(z) + \beta\phi_y(z) \\
&= \alpha f(x)(z) + \beta f(y)(z) \\
&= (\alpha f(x) + \beta f(y))(z),
\end{aligned} \tag{2.4.4}$$

for all $x, y, z \in V$, and $\alpha, \beta \in F$. Now $\phi_v = 0$ if and only if $v \in \mathrm{rad}_L(V)$. Since V is left-non-degenerate, $\mathrm{rad}_L(V) = \{0\}$. Therefore $f^{-1}\{0\} = \{0\}$, and f is injective. Since f is trivially surjective to $f(V)$, $f(V) \cong V$. Then $f(V) \cong V^*$ if and only if $V \cong V^*$, which is if and only if V is finite-dimensional by Theorem 2.2.21. ☐

Remark 2.4.4. In Theorem 2.4.3 a similar proof works for the right-non-degenerate case.

Theorem 2.4.5 (Orthogonal basis exists when finite-dimensional and non-degenerate). *Let V be a finite-dimensional non-degenerate symmetric bilinear space over F, with char$(F) \neq 2$. Then V has an orthogonal basis of non-null vectors.*

Proof. If V is trivial, then the claim is vacuously true; assume V is not trivial. Since V is non-degenerate, there exists $v \in V$ such that $v \cdot v \neq 0$ by Theorem 2.3.9. Let $S = \operatorname{span}(v)$. If $S = V$, then we are done. Assume $S \neq V$, and let $f : V \to S$ be such that

$$f(x) = \frac{x \cdot v}{v \cdot v} v. \tag{2.4.5}$$

The restriction of f to S is the identity; therefore the restriction is orthogonal. In addition, $f(V) = S$, $f^{-1}\{0\} = S^{\Vdash V}$, and f is linear and surjective. Now

$$\dim(V) = \dim(S) + \dim\big(S^{\Vdash V}\big) = \dim\big(S \dotplus S^{\Vdash V}\big) \tag{2.4.6}$$

by Theorem 2.2.12, and $V = S \dotplus S^{\Vdash V}$ by Theorem 2.2.17. Since in addition S and $S^{\Vdash V}$ are orthogonal by construction, $V = S \perp S^{\Vdash V}$. S and $S^{\Vdash V}$ are both non-degenerate by Theorem 2.3.6. By induction there exists an orthogonal basis $B \subset S^{\Vdash V}$ of non-null vectors. By construction, $\{v\} \cup B$ is then an orthogonal basis of non-null vectors. □

Theorem 2.4.6 (Orthogonal basis exists when finite-dimensional). *Let V be a finite-dimensional symmetric bilinear space over F, with char$(F) \neq 2$. Then V has an orthogonal basis $B \cup C \subset V$ such that span$(B) =$ rad(V), and the vectors in C are non-null.*

Proof. There is a non-degenerate subspace $W \subset V$ such that $V \cong W \perp \operatorname{rad}(V)$ by Theorem 2.3.2. Let $B \subset \operatorname{rad}(V)$ be any basis of $\operatorname{rad}(V)$; then B is automatically orthogonal. There is an orthogonal non-null basis $C \subset W$ of W by Theorem 2.4.5. By construction, $B \cup C$ is then an orthogonal basis of V. □

Theorem 2.4.7 (Orthogonal decomposition). *Let V be a finite-dimensional non-degenerate symmetric bilinear space over F, and $S \subset V$ be a subspace of V. Then $V \cong S \perp S^{\Vdash V}$ if and only if S is non-degenerate.*

Proof. Assume $V \cong S \perp S^{\Vdash V}$. Then S is non-degenerate by Theorem 2.3.6. Assume S is non-degenerate. Let $A \subset V$ be a non-null orthogonal basis of S, and $B \subset V$ be a non-null orthogonal basis of V, such that $A \subset B$; they exist by Theorem 2.4.5. Since for all $v \in A$ it holds that $v \cdot v \neq 0$, $v \notin S^{\Vdash V}$. Therefore $S \cap S^{\Vdash V} = \{0\}$. On the other hand, by construction every element of $B \setminus A$ is orthogonal to every element of S. Therefore $S^{\Vdash V} = \operatorname{span}(B \setminus A)$, and $V = S \perp S^{\Vdash V}$. □

Example 2.4.8. Let $V = \mathbb{R}^2$ with the non-degenerate symmetric bilinear form

$$(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 - y_1 y_2. \tag{2.4.7}$$

Then $S = \operatorname{span}((1, 1))$ is a degenerate subspace of V, and $S^{\Vdash V} = S$.

Theorem 2.4.9 (Non-degenerateness by orthogonal complement). *Let V be a finite-dimensional non-degenerate symmetric bilinear space over F, and $S \subset V$ be a subspace of V. Then $S^{\Vdash V}$ is non-degenerate if and only if S is non-degenerate.*

Proof. S is non-degenerate if and only if $V \cong S \perp S^{\Vdash V}$ by Theorem 2.4.7. This is equivalent to S and $S^{\Vdash V}$ being non-degenerate by Theorem 2.3.6. □

Theorem 2.4.10 (Double orthogonal complement is identity when finite-dimensional, non-degenerate, and symmetric). *Let V be a finite-dimensional non-degenerate symmetric bilinear space, and $S \subset V$ be a subspace of V. Then $S = S^{\Vdash V^{\Vdash V}}$ if and only if S is non-degenerate.*

Proof. Assume S is non-degenerate. Then

$$S \perp S^{\Vdash V} \cong V \cong S^{\Vdash V} \perp S^{\Vdash V^{\Vdash V}} \tag{2.4.8}$$

by Theorem 2.4.7. It follows that

$$\dim(S) + \dim\left(S^{\Vdash V}\right) = \dim\left(S^{\Vdash V}\right) + \dim\left(S^{\Vdash V^{\Vdash V}}\right) \tag{2.4.9}$$

by Theorem 2.2.17. Therefore $\dim\left(S^{\Vdash V^{\Vdash V}}\right) = \dim(S)$, and $S \cong S^{\Vdash V^{\Vdash V}}$ by Theorem 2.2.17. Therefore $S = S^{\Vdash V^{\Vdash V}}$. Assume $S = S^{\Vdash V^{\Vdash V}}$, and let $v \in \operatorname{rad}(S)$. Then $v \in S^{\Vdash S} \subset S^{\Vdash V}$, and also $v \in S = S^{\Vdash V^{\Vdash V}}$. Therefore $v \in S \cap S^{\Vdash V}$. TODO. □

2.5 Real bilinear spaces

Let V be a bilinear space over $\mathbb{R}$. A bilinear form in V, and the V itself, is called

- **positive-definite**, if $\forall x \in V \setminus \{0\} : x \cdot x > 0$,
- **negative-definite**, if $\forall x \in V \setminus \{0\} : x \cdot x < 0$,
- **definite**, if it is either positive-definite or negative-definite,
- **positive-semi-definite**, if $\forall x \in V \setminus \{0\} : x \cdot x \geq 0$,
- **negative-semi-definite**, if $\forall x \in V \setminus \{0\} : x \cdot x \leq 0$,
- **semi-definite**, if it is either positive-semi-definite or negative-semi-definite,
- **indefinite**, if it is not semi-definite, and
- *fulfilling the Cauchy-Schwarz inequality* , if $\forall x, y \in V : (x \cdot y)^2 \leq (x \cdot x)(y \cdot y)$.

If V is symmetric, then an orthogonal basis $B = \{b_i\}_{i \in I} \subset V$ of V is called **orthonormal** if $b_i \cdot b_i \in \{-1, 0, 1\}$, for all $i \in I$. Let $p, q, r \in \mathbb{N}$, and $n = p + q + r$. Then $\mathbb{R}^{p,q,r}$ is the vector space $\mathbb{R}^n$ over $\mathbb{R}$, together with a symmetric bilinear form $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ for which

$$e_i \cdot e_i = \begin{cases} 1, & \text{if } 1 \leq i \leq p, \\ -1, & \text{if } p < i \leq p + q, \text{ and} \\ 0, & \text{if } p + q < i \leq p + q + r, \end{cases}$$

where e_i is the i:th standard basis vector of $\mathbb{R}^n$. We denote $\mathbb{R}^{p,q} = \mathbb{R}^{p,q,0}$.

Example 2.5.1. An inner product in V is a real positive-definite symmetric bilinear form in V.

Theorem 2.5.2 (Orthonormal basis exists when finite-dimensional and symmetric). *Let V be a finite-dimensional symmetric bilinear space over $\mathbb{R}$. Then V has an orthonormal basis. If V is non-degenerate, then the basis can be chosen non-null.*

Proof. There exists an orthogonal basis $C = \{c_i\}_{i \in I}$ of V by Theorem 2.4.6. Let $B = \{b_i\}_{i \in I} \subset V$ be such that

$$b_i = \begin{cases} \frac{c_i}{\sqrt{|c_i \cdot c_i|}}, & \text{if } c_i \cdot c_i \neq 0, \\ c_i, & \text{if } c_i \cdot c_i = 0. \end{cases} \tag{2.5.1}$$

for all $i \in I$. Then B is an orthonormal basis of V. If V is non-degenerate, then C can be chosen non-null by Theorem 2.4.6, making B a non-null orthonormal basis of V. $\square$

Theorem 2.5.3 (Null is in radical when finite-dimensional, symmetric and semi-definite). *Let V be a finite-dimensional symmetric semi-definite bilinear space over $\mathbb{R}$, and $v \in V$. Then $v \cdot v = 0$ if and only if $v \in rad(V)$.*

Proof. Assume $v \in \text{rad}(V)$. Then in particular $v \cdot v = 0$. Assume $v \cdot v = 0$. There exists an orthonormal basis $B = \{b_i\}_{i \in I}$ of V by Theorem 2.5.2. Since V is semi-definite, $b_i \cdot b_i \in \{0, 1\}$, for all $i \in I$. Let $u \in V$. Since B generates V, there exists unique $\alpha, \beta \in \widehat{F^I}$ such that

$$\begin{aligned} u &= \sum_{i \in I} \alpha_i b_i, \\ v &= \sum_{j \in I} \beta_j b_j. \end{aligned} \tag{2.5.2}$$

Then

$$v \cdot v = \sum_{j \in I} \beta_j^2 (b_j \cdot b_j). \tag{2.5.3}$$

Since $v \cdot v = 0$, it holds that either $\beta_j = 0$, or $b_j \cdot b_j = 0$, for all $j \in I$. Now

$$\begin{aligned} u \cdot v &= \sum_{i \in I} \sum_{j \in I} \alpha_i \beta_j (b_i \cdot b_j) \\ &= \sum_{i \in I} \alpha_i \beta_i (b_i \cdot b_i) \\ &= 0. \end{aligned} \tag{2.5.4}$$

Therefore $v \in \text{rad}(V)$. $\square$

Theorem 2.5.4 (Cauchy-Schwarz is equivalent to semi-definitiness when finite-dimensional and symmetric). *Let V be a finite-dimensional symmetric bilinear space over $\mathbb{R}$. Then the Cauchy-Schwarz inequality holds if and only if V is semi-definite.*

Proof. Assume the Cauchy-Schwarz inequality holds but that V is indefinite. Then there exists $x, y \in V$ such that $x \cdot x > 0$ and $y \cdot y < 0$. Then $(x \cdot y)^2 \leq (x \cdot x)(y \cdot y)$ does not hold since the left-hand side is non-negative and the right-hand side is negative; a contradiction. Therefore V is semi-definite. Assume V is positive-semi-definite; the proof for the negative-semi-definite case is similar. If $y \cdot y = 0$, then $x \cdot y = 0$ by Theorem 2.5.3, and the Cauchy-Schwarz inequality holds. If $y \cdot y \neq 0$, then the Cauchy-Schwarz inequality can be modified to the following equivalent form:

$$\forall y \in V : |x_\parallel \cdot x_\parallel| \leq |x \cdot x|,$$

where $x_\parallel = \frac{x \cdot y}{y \cdot y} y$ is the orthogonal projection of x to y, and $x_\perp = x - x_\parallel$ is the rejection of x from y. Then

$$\begin{aligned}
x_\parallel \cdot x_\parallel &\leq x_\parallel \cdot x_\parallel + x_\perp \cdot x_\perp && (2.5.5)\\
&= x_\parallel \cdot x_\parallel + 2x_\perp \cdot x_\parallel + x_\perp \cdot x_\perp && (2.5.6)\\
&= (x_\parallel + x_\perp) \cdot (x_\parallel + x_\perp) && (2.5.7)\\
&= x \cdot x, && (2.5.8)
\end{aligned}$$

where we used the fact that $x_\parallel \cdot x_\perp = 0$. Since V is positive-semi-definite, $|x_\parallel \cdot x_\parallel| \leq |x \cdot x|$. Therefore the Cauchy-Schwarz inequality holds. □

Example 2.5.5. The bilinear form in $\mathbb{R}^{p,q,r}$ is positive-definite if $p = n$, negative definite if $q = n$, positive-semi-definite if $q = 0$, negative-semi-definite if $p = 0$, indefinite if both $p > 0$ and $q > 0$, and non-degenerate if $r = 0$.

Theorem 2.5.6 (Finite-dimensional bilinear spaces can be implemented on a computer). *Let V be a finite-dimensional symmetric bilinear space of signature (p, q, r) over $\mathbb{R}$. Then V is isomorphic to $\mathbb{R}^{p,q,r}$.*

Proof. Let $n = \dim(V)$. There exists an orthonormal basis $B = \{b_1, \dots, b_n\} \subset V$ by Theorem 2.5.2. This basis can be reordered such that

$$b_i \cdot b_i = \begin{cases} 1, & \text{if } 1 \leq i \leq p, \\ -1, & \text{if } p < i \leq p + q, \text{ and} \\ 0, & \text{if } p + q < i \leq p + q + r. \end{cases}$$

Let $\phi : V \to \mathbb{R}^{p,q,r}$ be a linear function such that $\forall i \in [1, n] : \phi(b_i) = e_i$, where $e_i \in \mathbb{R}^n$ is the i:th standard basis vector. Then ϕ is bijective, linear, and

$$\begin{aligned}
\phi(x) \cdot \phi(y) &= \phi\left(\sum_{i=1}^{n} x_i b_i\right) \cdot \phi\left(\sum_{j=1}^{n} y_j b_j\right) \\
&= \left(\sum_{i=1}^{n} x_i \phi(b_i)\right) \cdot \left(\sum_{j=1}^{n} y_j \phi(b_j)\right) \\
&= \left(\sum_{i=1}^{n} x_i e_i\right) \cdot \left(\sum_{j=1}^{n} y_j e_j\right) \\
&= \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i \\
&= \left(\sum_{i=1}^{n} x_i b_i\right) \cdot \left(\sum_{j=1}^{n} y_j b_j\right) \\
&= x \cdot y.
\end{aligned} \tag{2.5.9}$$

Therefore V is isomorphic to $\mathbb{R}^{p,q,r}$ as a bilinear space. $\square$

Remark 2.5.7. By Theorem 2.5.6, for computational purposes we may always concentrate on $\mathbb{R}^{p,q,r}$ without loss of generality, rather than on an abstract real vector space V with a symmetric bilinear form of signature (p, q, r).

2.6 Tensor product

Let $\{V_i\}_{i \in I}$ be a set of vector spaces over F, U_i be the free vector space on V_i, and $W = \bigoplus_{i \in I} U_i$. Let $S \subset W$ be a subspace of W defined by

$$S = \left\{ f \in W : \forall i \in I : \sum_{v \in V_i} f_i(v) v = 0 \right\}. \tag{2.6.1}$$

Then the **tensor product** of $\{V_i\}_{i \in I}$ is defined by

$$\bigotimes_{i \in I} V_i = W/S. \tag{2.6.2}$$

When $I = \emptyset$, the tensor product is defined to be F. In the common case where $V_i = V$ for all $i \in I$, the tensor product is denoted by $V^{\otimes I}$, and called the **I-fold tensor product**. The elements of $V^{\otimes I}$ are called **I-vectors**.

Remark 2.6.1 (Intuition for the tensor product). The free vector space U_i contains finite linear combinations $f_i \in U_i$ of the elements of V_i, encoded such that $f_i(v)$ gives the weight of the element $v \in V_i$. The linear combination f_i represents the element $\sum_{v \in V_i} f_i(v) v \in V_i$, which is valid since f_i is non-zero only at finitely many positions. If $f, g \in W$ and $f_i - g_i$ represents the zero vector for all $i \in I$ (that is, $f - g \in S$), then f and g represent the same vector in $\bigoplus_{i \in I} V_i$. These linear combinations are identified by the quotient W/S.

Remark 2.6.2 (Notation for the tensor product). Being a quotient vector space, the elements of the tensor product are equivalence classes. Notationally, however, we will work with their class representatives instead. We will demonstrate the notation for the tensor product $U \otimes V$, where U and V are vector spaces over $\mathbb{R}$. Let $[f] = [(f_U, f_V)] \in U \otimes V$, where the brackets denote the equivalence class of the class representative. Each linear combination is written as an explicit sum, as in $f_U \sim 2u_1 - 4u_2$, and $f_V \sim v$, where $u_1, u_2 \in U$, and $v \in V$. We group the linear combinations f_U and f_V together by juxtaposition, as in $f = (f_U, f_V) \sim (2u_1 - 4u_2)v$. Finally, we leave out the equivalence class brackets. It then makes sense to write an equation such as

$$(2u_1 - 4u_2)v = 2u_1v - 4u_2v. \tag{2.6.3}$$

This is similar to the use of rational numbers, where one writes equations of the form $1/1 = 2/2$.

Remark 2.6.3. The tensor product of a single vector space V is isomorphic to V.

Example 2.6.4. The elements of $V^{\otimes 2}$ are called 2-vectors.

2.7 Algebras

An **algebra over a field**, from now on simply an algebra, is a vector space V over F together with a bilinear function $\circledast : V^2 \to V$, called the **product** in V. The product is usually denoted by juxtaposition. The algebra, and the product, is called

- **associative**, if $\forall x, y, z \in V : (xy)z = x(yz)$,
- **commutative**, if $\forall x, y \in V : xy = yx$,
- **unital**, if $\exists e_0 \in V : \forall x \in W : e_0x = x = xe_0$, and

The element e_0 above, it it exists, is called the **identity element** of V. If for elements $x, y \in V$, with $x \neq 0$, there exists a unique element $r_L \in V$ such that $y = r_Lx$ and a unique element $r_R \in V$ such that $y = xr_R$, then V is called a **division algebra**. Let V be an associative unital algebra. If for an element $x \in V$ there exists an element $y \in V$ such that $xy = yx = e_0$, then y is called an **inverse** of x. For any algebra V, a **sub-algebra** of V is a subspace of V which is closed under the product operation. Let W be an algebra over F. The **external direct sum** $V \oplus W$ of algebras V and W is the external direct sum of V and W as vector spaces, together with the product $(V \oplus W)^2 \to V \oplus W$ defined by $(v_1, w_2)(v_2, w_2) \mapsto (v_1v_2, w_1w_2)$, for all $v_1, v_1 \in V$, and $w_1, w_2 \in W$. The **internal direct sum** of sub-algebras is defined similarly. The **tensor product** of algebras is the tensor product of vector spaces, together with a bilinear product which applies the component products element-wise, e.g. $(u_1 \otimes v_1)(u_2 \otimes v_2) \mapsto (u_1u_2) \otimes (v_1v_2)$. An **algebra homomorphism** is a linear function $f : V \to W$ which preserves multiplication, i.e.

$$f(xy) = f(x)f(y), \tag{2.7.1}$$

for all $x, y \in V$. The set of algebra homomorphisms from V to W is denoted by $\mathrm{Hom}(V, W)$. If algebras V and W are unital, then an algebra homomorphism is also required to preserve the identity element, i.e. if $f \in \mathrm{Hom}(V, W)$, then

$$f({e_0}^V) = {e_0}^W. \tag{2.7.2}$$

An **algebra anti-homomorphism** is a linear function $f : V \to W$ which preserves multiplication, but reverses its order, i.e.

$$f(xy) = f(y)f(x), \tag{2.7.3}$$

for all $x, y \in V$.

Example 2.7.1. A field is a commutative division algebra over itself.

Theorem 2.7.2 (Identity element is unique in an algebra). *If an algebra W has an identity element $e_0 \in W$, then it is unique.*

Proof. Assume $e_0, e_0' \in W$ are both identity elements of W. Then

$$e_0' = e_0' e_0 = e_0.$$

□

Theorem 2.7.3 (Inverses are unique in an associative unital algebra). *In an associative unital algebra W, inverses are unique.*

Proof. Assume $y, y' \in W$ are both inverses of $x \in W$. Then

$$y' = y'(xy) = (y'x)y = y.$$

□

Theorem 2.7.4 (Division algebra for associative unital algebras). *An associative unital algebra W is a division algebra if and only if every element in $W \setminus \{0\}$ has an inverse.*

Proof. Assume W is a division algebra, and $x \in W \setminus \{0\}$. Then in particular there exists unique $r_L, r_R \in W$ such that $e_0 = r_L x = x r_R$. Thus

$$r_L = r_L(x r_R) = (r_L x) r_R = r_R,$$

and $x^{-1} = r_L = r_R$. Therefore every element in $W \setminus \{0\}$ has an inverse. Assume every element in $W \setminus \{0\}$ has an inverse, and let $x, y \in W$, with $x \neq 0$. Then $y = y(x^{-1}x) = (yx^{-1})x$ and $y = (xx^{-1})y = x(x^{-1}y)$. Since by Theorem 2.7.3 inverses are unique in W, so are yx^{-1} and $x^{-1}y$. Therefore W is a division algebra. □

Theorem 2.7.5 (Sum of homomorphisms of unital algebras is not a homomorphism). *Let V and W be associative unital algebras, and $f, g \in Hom(V, W)$. Then $f + g \notin Hom(V, W)$.*

Proof. Now

$$(f + g)(1) = f(1) + g(1) = 1 + 1. \tag{2.7.4}$$

Since an algebra homomorphism must preserve the identity element, this implies $1+1 = 1$, which in any field is equivalent to $1 = 0$; a contradiction. Therefore $f + g$ is not a homomorphism. □

Theorem 2.7.6 (Frobenius theorem). *Let V be a finite-dimensional associative division-algebra over $\mathbb{R}$. Then V is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Remark 2.7.7. Since Clifford algebras are finite-dimensional associative algebras, Theorem 2.7.6 shows that those Clifford algebras not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ will necessarily contain zero divisors.

2.8 Tensor algebra

Let V be a vector space over F. The **tensor algebra** $T(V)$ of V is the direct sum

$$T(V) = \bigoplus_{k \in \mathbb{N}} V^{\otimes k} \tag{2.8.1}$$

as vector spaces, together with a bilinear **product** $T(V) \times T(V) \to T(V)$, denoted by juxtaposition, defined by

$$(a_1 \cdots a_k)(b_1 \cdots b_l) = a_1 \cdots a_k b_1 \cdots b_l, \tag{2.8.2}$$

where $\{a_1, \dots, a_k\} \subset V$, $\{b_1, \dots, b_l\} \subset V$, and $k, l \in \mathbb{N}$. The elements of $T(V)$ are called **multi-vectors**.

Remark 2.8.1 (Notation for tensor algebra). Building on the notation for tensor products, the elements of tensor algebra are written in the form $4 + 3u - (u + v)w$, where $u, v, w \in V$, as is usual for direct sums. While strictly speaking there are two different plus operators in this expression, this abuse of notation does not cause any confusion.

2.9 Magnitude

Let V be a vector space over $\mathbb{R}$. A function $f : V \to \mathbb{R}$ is called

- **positive-homogeneous**, if $\forall x \in V : \forall \alpha \in \mathbb{R} : f(\alpha x) = |\alpha| f(x)$,
- **additive**, if $\forall x, y \in V : f(x + y) = f(x) + f(y)$,
- **sub-additive**, if $\forall x, y \in V : f(x + y) \leq f(x) + f(y)$,
- **convex** if $\forall x, y \in V : \forall t \in [0, 1] \subset \mathbb{R} : f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$,
- **quasi-convex**, if $\forall x, y \in V : \forall t \in [0, 1] \subset \mathbb{R} : f((1-t)x+ty) \leq \max\{f(x), f(y)\}$, and
- **positive-definite**, if $\forall x \in V : f(x) \geq 0$ and $f(x) = 0 \Leftrightarrow x = 0$.

A **semi-magnitude** in V is a function $f : V \to \mathbb{R}$ such that it is non-negative and positive-homogeneous. A **magnitude** in V is a positive-definite semi-magnitude in V. A **semi-norm** in V is a sub-additive semi-magnitude in V. A **norm** V is a positive-definite semi-norm in V. If V is actually an algebra over $\mathbb{R}$, then the f is called

- **multiplicative**, if $\forall x, y \in V : f(xy) = f(x)f(y)$,
- **sub-multiplicative**, if $\forall x, y \in V : f(xy) \leq f(x)f(y)$.

Remark 2.9.1. Sub-additivity is also known as the triangle inequality.

Remark 2.9.2. The definitions for magnitudes are my own.

Theorem 2.9.3 (Convexity is sub-additivity when positive-homogeneous). *Let $f : V \to \mathbb{R}$ be a positive-homogeneous function. Then f is convex if and only if it is sub-additive.*

Proof. Assume f is convex. Then

$$\begin{aligned} f(x+y) &= 2f(0.5x+0.5y) && (2.9.1)\\ &\leq 2(f(0.5x)+f(0.5y)) && (2.9.2)\\ &= f(x)+f(y). && (2.9.3) \end{aligned}$$

Therefore f is sub-additive. Assume f is sub-additive. Let $t \in [0,1] \in \mathbb{R}$. Then

$$\begin{aligned} f((1-t)x+ty) &\leq f((1-t)x)+f(ty) && (2.9.4)\\ &= (1-t)f(x)+tf(y). && (2.9.5) \end{aligned}$$

Therefore f is convex. □

Theorem 2.9.4 (Convexity implies quasi-convexity). *Let $f : V \to \mathbb{R}$ be a convex function. Then f is quasi-convex.*

Proof. Assume f is convex. Let $x, y \in V$, and $t \in [0,1] \subset \mathbb{R}$. Then

$$\begin{aligned} f((1-t)x+tf(y)) &\leq (1-t)f(x)+tf(y) && (2.9.6)\\ &\leq (1-t)\max\{f(x),f(y)\}+t\max\{f(x),f(y)\} && (2.9.7)\\ &= \max\{f(x),f(y)\}. && (2.9.8) \end{aligned}$$

Therefore f is quasi-convex. □

Theorem 2.9.5 (Quasi-convexity implies convexity when non-negative positive-homogeneous). *Let $f : V \to \mathbb{R}$ be a non-negative positive-homogeneous quasi-convex function. Then f is convex.*

Proof. For $x, y \in V$, let

$$t = \frac{f(y)}{f(x)+f(y)}.$$

Then

$$\begin{aligned} f\left(\frac{x+y}{f(x)+f(y)}\right) &= f\left((1-t)\frac{x}{f(x)}+t\frac{y}{f(y)}\right) && (2.9.9)\\ &\leq \max\left\{f\left(\frac{x}{f(x)}\right), f\left(\frac{y}{f(y)}\right)\right\} && (2.9.10)\\ &= 1. && (2.9.11) \end{aligned}$$

Using positive-homogenuity and non-negativeness,

$$f(x+y) \leq f(x)+f(y).$$

Thus f is sub-additive. By Theorem 2.9.3 this is equivalent to the convexity of f. Therefore f is convex. □

Remark 2.9.6 (Quasi-convexity is equivalent to sub-level sets being convex). Quasi-convexity of $f : V \to \mathbb{R}$ is equivalent to the sub-level sets of f being convex, which gives an intuitive understanding of a semi-norm. In general quasi-convexity of f is not equivalent to the convexity of f; for non-negative positive-homogeneous functions f it is.

Theorem 2.9.7 (Semi-definiteness is sub-additivity when symmetric). *Let $\cdot : V^2 \to \mathbb{R}$ be a symmetric bilinear form, and $\|\cdot\| : V \to \mathbb{R} : \|x\| = \sqrt{|x \cdot x|}$. Then $\cdot$ is semi-definite if and only if $\|\cdot\|$ is sub-additive.*

Proof. Assume $\cdot$ is indefinite. By indefiniteness there exists $x, y \in V$ such that $x \cdot x > 0$ and $y \cdot y < 0$. Now one can solve the quadratic equation $\|(1-t)x + ty\| = 0$ for $t \in \mathbb{R}$. The solution is

$$t = \frac{x \cdot (x+y) \pm \sqrt{(x \cdot y)^2 - (x \cdot x)(y \cdot y)}}{(x+y) \cdot (x+y)}.$$

The discriminant is always positive, since $(x \cdot x)(y \cdot y) < 0$. Therefore, there are two points $a, b \in V$, with $\|a\| = 0$ and $\|b\| = 0$, which lie on the same line as x and y. Either x or y is a convex combination of a and b. Without loss of generality, assume it is x. If $\|\cdot\|$ were convex, it would hold that $\|x\| = 0$. Since this is not the case, $\|\cdot\|$ is not convex. By Theorem 2.9.3 this is equivalent to $\|\cdot\|$ not being sub-additive. Assume $\cdot$ is semi-definite. Then the Cauchy-Schwarz inequality holds by Theorem 2.5.4. By semi-definiteness $(x \cdot x)(y \cdot y) \geq 0$. Now

$$\begin{aligned}
\|x+y\|^2 &= |(x+y) \cdot (x+y)| && (2.9.12)\\
&= |x \cdot x + 2x \cdot y + y \cdot y| && (2.9.13)\\
&\leq |x \cdot x| + 2|x \cdot y| + |y \cdot y| && (2.9.14)\\
&\leq |x \cdot x| + 2\sqrt{(x \cdot x)(y \cdot y)} + |y \cdot y| && (2.9.15)\\
&= |x \cdot x| + 2\sqrt{|x \cdot x|}\sqrt{|y \cdot y|} + |y \cdot y| && (2.9.16)\\
&= (\|x\| + \|y\|)^2. && (2.9.17)
\end{aligned}$$

Thus $\|\cdot\|$ is sub-additive. □

Theorem 2.9.8 (Norm from a symmetric bilinear form). *Let $\cdot$ be a symmetric bilinear form in $\mathbb{R}$. Then $\|\cdot\| : V \to \mathbb{R} : \|x\| = \sqrt{|x \cdot x|}$ is a semi-norm (norm) if and only if $\cdot$ is semi-definite (definite).*

Proof. Clearly $\|\cdot\|$ is non-negative. Homogenuity is shown by

$$\begin{aligned}
\|\alpha x\|^2 &= |(\alpha x) \cdot (\alpha x)| && (2.9.18)\\
&= \alpha^2 |x \cdot x| && (2.9.19)\\
&= \alpha^2 \|x\|^2. && (2.9.20)
\end{aligned}$$

where $\alpha \in \mathbb{R}$, and $x \in V$. By Theorem 2.9.7, $\cdot$ is sub-additive if and only if it is semi-definite. The $\cdot$ is definite if and only if $\|\cdot\|$ is positive-definite. □

2.10 Topological spaces

Let X be a set. A set $T_X \subset \mathcal{P}(X)$ is called a **topology** on X, if

- $\emptyset \in T_X$,
- $O_1 \cap O_2 \in T_X$, for all $O_1, O_2 \in T_X$,
- $\bigcup_{i \in I} O_i \in T_X$, for all $\{O_i\}_{i \in I} \subset T_X$,

- $X \in T_X$.

A **topological space** is a set X together with a topology T_X on X. An element $O \in T_X$ is called an **open set** of X. Let $S \subset X$. A **neighborhood** of S in X is an open set $O_S \in T_X$ such that $S \subset O_S$. The set of neighborhoods of S in X is denoted by $T_X(S)$. If $p \in X$, then we will abbreviate $T_X(p) = T_X(\{p\})$. A **closed set** of X is an element of

$$C_X = \{X \setminus U : U \in T_X\}. \tag{2.10.1}$$

The set of closed sets containing S is denoted by $C_X(S)$. The **closure** of S on X is defined by

$$\overline{S} = \bigcap C_X(S). \tag{2.10.2}$$

A subset $B_X \subset T_X$ is called a **basis** of X, if

$$T_X = \left\{\bigcup S : S \subset B_X\right\}. \tag{2.10.3}$$

In this case we also say that B_X *generates* T_X. A subset $B_X(p) \subset T_X(p)$ is called a **neighborhood basis** at $p \in X$, if

$$\forall O_p \in T_X(p) : \exists O'_p \in B_X(p) : O'_p \subset O_p. \tag{2.10.4}$$

Let Y be a topological space. The **product space** $X \times Y$ of X and Y is the set $X \times Y$, together with a topology generated by

$$B_{X \times Y} = \{O \times N : O \in T_X, N \in T_Y\}, \tag{2.10.5}$$

called the **product topology**. Let X and Y be topological spaces, $f : X \to Y$ be a function, $p \in X$, and $y \in Y$. Then y is a **limit** of f at p, denoted $\lim_{x \to p} f(x) = y$, if

$$\forall O_y \in T_Y(y) : \exists O_p \in T_X(p) : f(O_p \setminus \{p\}) \subset O_y. \tag{2.10.6}$$

A function $f : X \to Y$ is called **continuous** at $p \in X$, if

$$\lim_{x \to p} f(x) = f(p), \tag{2.10.7}$$

and **continuous**, if it is continuous for all $p \in X$.

Remark 2.10.1 (Being open or closed is not exclusive). Subsets of X may be open, closed, open and closed, or neither open or closed. For example, $\emptyset$ and X are always both open and closed.

Theorem 2.10.2 (Continuity by open sets). *Let X and Y be topological spaces, and $f : X \to Y$ be a function. Then f is continuous if and only if $f^{-1}(O) \in T_X$, for all $O \in T_Y$.*

Theorem 2.10.3 (Continuity is local). *Let X and Y be topological spaces, and $f : X \to Y$ be a function. Then f is continuous if and only if f is continuous at p for all $p \in X$.*

Theorem 2.10.4 (Limit by neighborhood bases). *Let X and Y be topological spaces, $B_X(p)$ be a neighborhood basis at $p \in X$, $B_Y(y)$ be a neighborhood basis at $y \in Y$, and $f : X \to Y$ be a function. Then $\lim_{x \to p} f(x) = y$ if and only if*

$$\forall N_y \in B_Y(y) : \exists O_p \in B_X(p) : f(O_p \setminus \{p\}) \subset N_y. \tag{2.10.8}$$

Proof. Assume $\lim_{x \to p} f(x) = y$, and let $N_y \in B_Y(y)$. Then there exists $O_p \in T_X(p)$ such that $f(O_p \setminus \{p\}) \subset N_y$. Since $B_X(p)$ is a neighborhood basis at p, there exists $O'_p \in B_X(p)$ such that $O'_p \subset O_p$. Now $f(O'_p \setminus \{p\}) \subset N_y$, and the result holds. Assume the formula holds. Let $N_y \in T_Y(y)$. Since $B_Y(y)$ is a neighborhood basis, there exists $N'_y \in B_Y(y)$ such that $N'_y \subset N_y$. Then by the formula there exists $O_p \in B_X(p) \subset T_X(p)$ such that $f(O_p \setminus \{p\}) \subset N'_y \subset N_y$. Therefore $\lim_{x \to p} f(x) = y$. □

Theorem 2.10.5 (Moving limit in). *Let X, Y, and Z be topological spaces, $p \in X$, $f : X \to Y$ be such that $\lim_{x \to p} f(x) = y$, and $g : Y \to Z$ be continuous at y. Then*

$$\lim_{x \to p} g(f(x)) = g(y). \tag{2.10.9}$$

Proof. Let $O_{g(y)} \in T_Z(g(y))$. Since g is continuous at y, there exists $N_y \in T_Y(y)$ such that $g(N_y) \subset O_{g(y)}$. Since $\lim_{x \to p} f(x) = y$, there exists $M_p \in T_X(p)$ such that $f(M_p \setminus \{p\}) \subset N_y$. Therefore $g(f(M_p \setminus \{p\})) \subset O_{g(y)}$. □

2.11 Topological vector spaces

A **topological vector space** is a vector space V together with a topology on V, such that addition and multiplication are continuous functions. The continuous linear functions are the homomorphisms of topological vector spaces.

Theorem 2.11.1 (Linear is continuous with finite-dimensional domain). *Let U be a finite-dimensional topological vector space, V be a topological vector space, and $f : U \to V$ be a linear function. Then f is continuous.*

Theorem 2.11.2 (Continuity at a point is continuity for linear functions). *Let U and V be topological vector spaces, and $f : U \to V$ be a linear function. Then f is continuous if and only if f is continuous at $p \in U$.*

Proof. Assume f is continuous. Then f is continuous at p by definition. Assume f is continuous at p, and let $q \in U$. Let $N_{f(q)} \in T_V(f(q))$. Then $N_{f(p)} = N_{f(q)} + (f(p) - f(q)) \in T_V(f(p))$, since addition by constant is a homeomorphism. Since f is continuous at p, there exists $O_p \in T_U(p)$ such that $f(O_p) \subset N_{f(p)}$. Then $O_q = O_p + (q - p) \in T_U(q)$ is such that

$$\begin{aligned} f(O_q) &= f(O_p + (q - p)) \\ &= f(O_p) + (f(q) - f(p)) \\ &\subset N_{f(p)} + (f(q) - f(p)) \\ &= N_{f(q)}. \end{aligned} \tag{2.11.1}$$

Therefore f is continuous at q. Since this holds for all $q \in U$, f is continuous. □

Theorem 2.11.3 (Sum of limits is the limit of sum). *Let X be a topological space, V be a Hausdorff topological vector space over a topological field F, $p \in X$, and $f, g : X \to V$ be functions, such that the limit of f at p exists, and the limit of g at p exists. Then*

$$\lim_{x \to p}[f(x) + g(x)] = \lim_{x \to p} f(x) + \lim_{x \to p} g(x). \tag{2.11.2}$$

Proof. Let $u = \lim_{x \to p} f(x)$, and $v = \lim_{x \to p} g(x)$. Let $N_{u+v} \in T_V(u + v)$. Since V is a topological vector space, addition is continuous. Therefore $+^{-1}(N_{u+v}) \in T_{V^2}$. Because of the product topology on V^2, there exists $N_u \in T_V(u)$, and $N_v \in T_V(v)$ such that $N_u \times N_v \subset +^{-1}(N_{u+v})$. By the definition of u and v, there exists $O_p \in T_X(p)$ such that $f(O_p \setminus \{p\}) \subset N_u$, and $g(O_p \setminus \{p\}) \subset N_v$. Therefore $f(O_p \setminus \{p\}) + g(O_p \setminus \{p\}) \subset N_{u+v}$. □

Theorem 2.11.4 (Scaling commutes with limits). *Let X be a topological space, V be a Hausdorff topological vector space over a topological field F, $p \in X$, $\alpha \in F$, and $f : X \to V$ be a function. Then the limit of f at p exists if and only if the limit of αf at p exists, and*

$$\lim_{x \to p} \alpha f(x) = \alpha \lim_{x \to p} f(x). \tag{2.11.3}$$

Proof. Assume $\lim_{x \to p} f(x)$ exists. Let $g : V \to V$ be such that $g(u) = \alpha u$. Since V is a topological vector space, multiplication is continuous, and then so is g. The result follows from Theorem 2.10.5. Assume $\lim_{x \to p} \alpha f(x)$ exists. The result holds trivially for $\alpha = 0$. Assume $\alpha \neq 0$. Then by the previous result

$$\begin{aligned} \frac{1}{\alpha} \lim_{x \to p} (\alpha f)(x) &= \lim_{x \to p} \frac{1}{\alpha} (\alpha f)(x) \\ &= \lim_{x \to p} f(x). \end{aligned} \tag{2.11.4}$$

□

2.12 Norm spaces

A **norm space** is a vector space V over a normed field F, together with a norm in V. An **open ball** in V is defined by

$$B_V(p, \delta) = \{v \in V : \|v - p\| < \delta\}, \tag{2.12.1}$$

and a **closed ball** in V is defined by

$$\overline{B}_V(p, \delta) = \{v \in V : \|v - p\| \leq \delta\}, \tag{2.12.2}$$

where $p \in V$, and $\delta \in \mathbb{R}$. The set of open balls at p is denoted by $B_V(p)$, and the set of closed balls at p is denoted by $\overline{B}_V(p)$. A **sphere** in V is defined by

$$S_V(p, \delta) = \{v \in V : \|v - p\| = \delta\}, \tag{2.12.3}$$

where $p \in V$, and $\delta \in \mathbb{R}$. A set $S \subset V$ is called **bounded**, if S is contained in some (open or closed) ball. A topology, called the **norm topology**, is defined on V by generating it from the open balls of U. Let U be a norm space, and $f : U \to V$ be a linear function. Then the **operator norm** of f is defined by

$$\|f\| = \sup_{u \in U \setminus \{0\}} \frac{\|f(u)\|}{\|u\|}. \tag{2.12.4}$$

The f is called **bounded**, if $\|f\| < \infty$.

Remark 2.12.1. The open balls at $p \in V$ form a neighborhood basis at p. Thus the notation $B_V(p)$ is consistent with neighborhood bases.

Remark 2.12.2. A closed ball $\overline{B}_V(p, \delta)$ is the closure of $B_V(p, \delta)$. Thus the notation for closed balls is consistent with closures.

Theorem 2.12.3 (Reverse triangle inequality). *Let U be a norm space over F, and $u, v \in U$. Then*

$$|\|u\| - \|v\|| \leq \|u - v\|. \tag{2.12.5}$$

Proof. By the sub-additivity of the norm

$$\begin{aligned} \|u\| - \|v\| &= \|u - v + v\| - \|v\| \\ &\leq \|u - v\| + \|v\| - \|v\| \\ &= \|u - v\|. \end{aligned} \tag{2.12.6}$$

Similarly, $\|v\| - \|u\| \leq \|v - u\|$. □

Theorem 2.12.4 (Norm is continuous in a norm space). *Let V be a norm space over F. Then the norm in V is continuous.*

Proof. This is shown by Theorem 2.12.3. □

Theorem 2.12.5 (All finite-dimensional norms are continuous). *Let V be a finite-dimensional norm space, and let $\|\cdot\|_b : V^2 \to \mathbb{R}$ be another norm in V. Then $\|\cdot\|_b$ is continuous.*

Proof. □

Theorem 2.12.6 (Scalar multiplication is continuous in a norm space). *Let V be a norm space over F. Then scalar multiplication is a continuous function $F \times V \to V$.*

Proof. Let $\epsilon \in \mathbb{R}_+$, $u, v \in V$, and $\alpha, \beta \in F$. We will show that scalar multiplication is continuous at (α, u). By the sub-additivity of the norm

$$\begin{aligned} \|\alpha u - \beta v\| &= \|\alpha u - \beta u + \beta u - \beta v\| \\ &= \|(\alpha - \beta)u + \beta(u - v)\| \\ &\leq |\alpha - \beta|\|u\| + |\beta|\|u - v\|. \end{aligned} \tag{2.12.7}$$

Let β be such that

$$|\alpha - \beta|\|u\| < \frac{\epsilon}{2}, \tag{2.12.8}$$

and let v be such that

$$|\beta|\|u-v\| < \frac{\epsilon}{2}, \tag{2.12.9}$$

which are possible since norm are continuous by Theorem 2.12.4. Then

$$\begin{aligned} \|\alpha u - \beta v\| &\leq |\alpha - \beta|\|u\| + |\beta|\|u-v\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{2.12.10}$$

Therefore scalar multiplication is continuous at (α, u). Since this holds for every point, addition is continuous by Theorem 2.10.3. □

Theorem 2.12.7 (Addition is continuous in a norm space). *Let V be a norm space over F. Then addition is a continuous function $V^2 \to V$.*

Proof. Let $\epsilon \in \mathbb{R}_+$, $u_1, u_2, v_1, v_2 \in V$. We will show that addition is continuous at $(u_1, v_1) \in V^2$. By the sub-additivity of the norm

$$\begin{aligned} \|(u_1+v_1)-(u_2+v_2)\| &= \|(u_1-u_2)+(v_1-v_2)\| \\ &\leq \|u_1-u_2\| + \|v_1-v_2\|. \end{aligned} \tag{2.12.11}$$

Let u_2 and v_2 be such that

$$\begin{aligned} \|u_1-u_2\| &< \frac{\epsilon}{2}, \\ \|v_1-v_2\| &< \frac{\epsilon}{2}, \end{aligned} \tag{2.12.12}$$

which is possible since the norm is continuous by Theorem 2.12.4. Then

$$\begin{aligned} \|(u_1+v_1)-(u_2+v_2)\| &\leq \|u_1-u_2\| + \|v_1-v_2\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{2.12.13}$$

Therefore addition is continuous at (u_1, v_1). Since this holds for every point, addition is continuous by Theorem 2.10.3. □

Theorem 2.12.8 (Norm space is a topological vector space). *Let V be a norm space over F. Then V is a topological vector space.*

Proof. Addition in V is continuous by Theorem 2.12.7, and scalar multiplication in V is continuous by Theorem 2.12.6. Therefore V is a topological vector space. □

Theorem 2.12.9 (Spheres are closed and bounded). *Let V be a norm space over F. Then a sphere in V is closed and bounded.*

Proof. Let $p \in V$, and $\delta \in \mathbb{R}$. Now

$$\begin{aligned} S_V(p,\delta) &= \overline{B}_V(p,\delta) \setminus B_V(p,\delta) \\ &= \overline{B}_V(p,\delta) \cap (V \setminus B_V(p,\delta)) \end{aligned} \tag{2.12.14}$$

is closed in V, since a closed ball is closed in V, an open ball is open in V with a closed complement, and since the intersection of closed sets is closed. Since $S_V(p,\delta)$ is contained in $\overline{B}_V(p,\delta)$, a sphere is also bounded. □

Theorem 2.12.10 (Finite-dimensional norms are equivalent). *Let V be a finite-dimensional norm space, and let $\|\cdot\|_b : V^2 \to \mathbb{R}$ be another norm in V. Then $\|\cdot\|$ and $\|\cdot\|_b$ are equivalent.*

Proof. The unit sphere $S_V(0,1)$ is closed and bounded in V by Theorem 2.12.9, and compact in V by the Heine-Borel theorem. The $\|\cdot\|_b$ is a continuous function by Theorem 2.12.5; so is its restriction to $S_V(0,1)$. Then there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha &= \min_{\|u\|=1} \|u\|_b, \\ \beta &= \max_{\|u\|=1} \|u\|_b \end{aligned} \tag{2.12.15}$$

by the extreme value theorem. Therefore

$$\alpha \leq \left\| \frac{v}{\|v\|} \right\|_b \leq \beta, \tag{2.12.16}$$

for all $v \in V \setminus \{0\}$. It follows that

$$\alpha\|v\| \leq \|v\|_b \leq \beta\|v\|, \tag{2.12.17}$$

for all $v \in V$ (the case $v = 0$ holds trivially). □

Theorem 2.12.11 (Finite-dimensional norm topology is unique). *Let V be a finite-dimensional norm space. Then the norm topology of V does not depend on the norm.*

Proof. Since any two norms on a finite-dimensional vector space are equivalent by Theorem 2.12.10, they agree on which sets are open and which are not. □

Theorem 2.12.12 (Alternative forms for the operator norm). *Let U and V be norm spaces over F, $f : U \to V$ be a linear function, and $S \subset \mathbb{R} \setminus \{0\}$, $S \neq \emptyset$. Then*

$$\|f\| = \sup_{\|u\| \in S} \frac{\|f(u)\|}{\|u\|}. \tag{2.12.18}$$

Proof. Since the norm is positive-homogeneous, and f is linear,

$$\frac{\|f(u)\|}{\|u\|} = \left\| \frac{f(u)}{\|u\|} \right\| = \left\| f\left(\frac{u}{\|u\|} \right) \right\|, \tag{2.12.19}$$

for all $u \in U \setminus \{0\}$. Therefore the value of $\frac{\|f(u)\|}{\|u\|}$ is independent of $\|u\|$, and we may as well concentrate on u such that $\|u\| \in S$. □

Example 2.12.13. Some reoccurring forms for the operator norm include

$$\begin{aligned} \|f\| &= \sup_{\|u\|=1} \|f(u)\| \\ &= \sup_{0<\|u\|<\delta} \frac{\|f(u)\|}{\|u\|} \\ &= \sup_{\|u\|>\delta} \frac{\|f(u)\|}{\|u\|}, \end{aligned} \tag{2.12.20}$$

where $\delta \in \mathbb{R}_+$.

Theorem 2.12.14 (Continuous is bounded for linear functions). *Let U and V be norm spaces over F, and $f : U \to V$ be a linear function. Then f is continuous if and only if it is bounded.*

Proof. Assume f is continuous. Since a norm space is a topological vector space by Theorem 2.12.8, and f is linear, this is equivalent to f being continuous at 0 by Theorem 2.11.2. Therefore there exists $\delta \in \mathbb{R}_+$ such that

$$\|f(u) - f(0)\| = \|f(u)\| < 1, \tag{2.12.21}$$

for all $u \in U$ such that $\|u\| < \delta$. Then

$$\begin{aligned}\frac{\|f(u)\|}{\|u\|} &= \left\| f\left(\frac{u}{\|u\|}\right) \right\| \\ &= \left\| \frac{1}{\delta} f\left(\delta \frac{u}{\|u\|}\right) \right\| \\ &= \frac{1}{|\delta|} \left\| f\left(\delta \frac{u}{\|u\|}\right) \right\| \\ &< \frac{1}{|\delta|},\end{aligned} \tag{2.12.22}$$

for all $u \in U$ such that $0 < \|u\| < \delta$. Therefore f is bounded by Theorem 2.12.12. Assume f is bounded. Then

$$\begin{aligned}\|f(u) - f(0)\| &= \|f(u)\| \\ &\leq \|f\|\|u\|,\end{aligned} \tag{2.12.23}$$

for all $u \in U \setminus \{0\}$. If $\epsilon \in \mathbb{R}_+$, then letting $\|u - 0\| = \|u\| < \frac{\epsilon}{\|f\|}$ shows that f is continuous at 0. Therefore f is continuous. □

2.13 Total derivative

Let U and V be norm spaces over $\mathbb{R}$, $S \subset U$, $p \in S$, and $f : S \to V$ be a function. The **differencing** at p is a partial function $\Delta_p : V^S \to V^{S-p}$ such that

$$\Delta_p(f)(u) = f(p + u) - f(p). \tag{2.13.1}$$

The $\Delta_p(f)$ is called the **difference** of f at p. The **differentiation** at p is a partial function $\mathcal{D}_p : V^S \to V^{S-p}$ such that $\mathcal{D}_p(f)$ is a continuous linear function, and

$$\lim_{u \to 0} \frac{\|\mathcal{D}_p(f)(u) - \Delta_p(f)(u)\|}{\|u\|} = 0 \tag{2.13.2}$$

in the subspace topology of $S - p$. The f is called **differentiable** at p, if $\mathcal{D}_p(f)$ exists, and **differentiable**, if f is differentiable at p for every $p \in S$. If f is differentiable at p, then the $\mathcal{D}_p(f)$ is called the **total derivative** of f at p.

Remark 2.13.1. The total derivative is also known as the **Fréchet derivative**.

Remark 2.13.2. The subset $S \subset U$ in the definition of the total derivative is arbitrary. In particular, it need not be open in U.

Theorem 2.13.3 (Total derivative is unique). *Let U and V be norm spaces over $\mathbb{R}$, $S \subset U$, $p \in S$, and $f : S \to V$ be differentiable at p. Then $\mathcal{D}_p(f)$ is unique.*

Proof. Let $\mathcal{D}_p(f)_1$ and $\mathcal{D}_p(f)_2$ both be total derivatives of f at p. By the definition of the limit, for every $\epsilon \in \mathbb{R}_+$ there exists $\delta \in \mathbb{R}_+$ such that

$$\begin{aligned} \|\Delta_p(f)(u) - \mathcal{D}_p(f)_1(u)\| &< \|u\|\epsilon/2, \\ \|\Delta_p(f)(u) - \mathcal{D}_p(f)_2(u)\| &< \|u\|\epsilon/2, \end{aligned} \tag{2.13.3}$$

for all $u \in S$ such that $0 < \|u\| < \delta$. By the sub-additivity of the norm,

$$\begin{aligned} \|\mathcal{D}_p(f)_2(u) - \mathcal{D}_p(f)_1(u)\| &= \|[\Delta_p(f)(u) - \mathcal{D}_p(f)_1(u)] - [\Delta_p(f)(u) - \mathcal{D}_p(f)_2(u)]\| \\ &\leq \|\Delta_p(f)(u) - \mathcal{D}_p(f)_1(u)\| + \|\Delta_p(f)(u) - \mathcal{D}_p(f)_2(u)\| \\ &< \|u\|\epsilon, \end{aligned} \tag{2.13.4}$$

for all $u \in S$ such that $0 < \|u\| < \delta$. Now

$$\begin{aligned} \|\mathcal{D}_p(f)_2 - \mathcal{D}_p(f)_1\| &= \sup_{0<\|u\|<\delta} \frac{\|\mathcal{D}_p(f)_2(u) - \mathcal{D}_p(f)_1(u)\|}{\|u\|} \\ &< \sup_{0<\|u\|<\delta} \frac{\|u\|\epsilon}{\|u\|} \\ &= \epsilon. \end{aligned} \tag{2.13.5}$$

Since this holds for all $\epsilon \in \mathbb{R}_+$, $\mathcal{D}_p(f)_2 = \mathcal{D}_p(f)_1$. □

Theorem 2.13.4 (Restriction of differentiable is differentiable). *Let U and V be norm spaces over $\mathbb{R}$, $p \in S' \subset S \subset U$, and $f : S \to V$ be differentiable at p. Then $f|S'$ is differentiable at p.*

Proof. Since f is differentiable at p, for every $\epsilon \in \mathbb{R}_+$ there exists $\delta \in \mathbb{R}_+$ such that

$$\|\mathcal{D}_p(f)_2(u) - \mathcal{D}_p(f)_1(u)\| < \|u\|\epsilon, \tag{2.13.6}$$

for all $u \in S$ such that $0 < \|u\| < \delta$. This claim still holds if we require in addition that $u \in S'$. □

Remark 2.13.5. Note that in Theorem 2.13.4 the restriction is to an arbitrary subset S'. In particular, S' need not be open in S. If $S' = \{p\}$, then $f|S'$ is vacuously differentiable at p.

Example 2.13.6. The converse of Theorem 2.13.4 does not hold in general for arbitrary subsets $S' \subset S$. For example, let $f : \mathbb{R} \to \mathbb{R} : f(x) = |x|$, and $S' = [0, 1] \subset \mathbb{R}$. Then $f|S'$ is differentiable at 0, but f is not differentiable at 0.

Theorem 2.13.7 (Differentiability is local). *Let U and V be norm spaces over $\mathbb{R}$, $p \in S' \subset S \subset U$, $S' \in T_S(p)$, and $f : S \to V$. Then f is differentiable at p if and only if $f|S'$ is differentiable at p.*

Proof. Assume f is differentiable at p. Then Theorem 2.13.4 shows that $f|S'$ is differentiable at p. Assume $f|S$ is differentiable at p. Since S' is open in S, the topology of S' consists exactly of those open sets of S which are contained in S'. Therefore the differentiability of $f|S'$ at p implies the differentiability of f at p. □

Theorem 2.13.8 (Total derivative of a constant is zero). *Let U and V be norm spaces over $\mathbb{R}$, $p \in U$, $S \in T_U(p)$, and $f : S \to V$ be a constant function. Then f is differentiable at p, and $\mathcal{D}_p(f) = 0$.*

Proof. Let $u \in S \setminus \{0\}$. Then

$$\frac{\|\Delta_p(f)(u) - 0\|}{\|u\|} = 0. \tag{2.13.7}$$

Therefore $\mathcal{D}_p(f) = 0$, since the total derivative is unique by Theorem 2.13.3. □

Theorem 2.13.9 (Scaling rule for total derivative). *Let U and V be norm spaces over $\mathbb{R}$, $f : U \to V$ be differentiable at $p \in U$, and $\alpha \in \mathbb{R}$. Then f is differentiable at p if and only if αf is differentiable at p, and*

$$\mathcal{D}_p(\alpha f) = \alpha \mathcal{D}_p(f). \tag{2.13.8}$$

Proof. Assume f is differentiable at p, and $u \in U \setminus \{0\}$. Then

$$\begin{aligned} \frac{\|\Delta_p(\alpha f) - \alpha \mathcal{D}_p(f)\|}{\|u\|} &= \frac{\|\alpha \Delta_p(f) - \alpha \mathcal{D}_p(f)\|}{\|u\|} \\ &= |\alpha| \frac{\|\Delta_p(f) - \mathcal{D}_p(f)\|}{\|u\|}. \end{aligned} \tag{2.13.9}$$

Taking the limit $u \to 0$ on both sides gives

$$\lim_{u \to 0} \frac{\|\Delta_p(\alpha f) - \alpha \mathcal{D}_p(f)\|}{\|u\|} \leq 0, \tag{2.13.10}$$

since f is differentiable at p, and by Theorem 2.11.4. The result follows since total derivative is unique by Theorem 2.13.3. Assume αf is differentiable at p. The result is trivial for $\alpha = 0$. Assume $\alpha \neq 0$. Then by the previous result

$$\begin{aligned} \frac{1}{\alpha} \mathcal{D}_p(\alpha f) &= \mathcal{D}_p\left(\frac{1}{\alpha} \alpha f\right) \\ &= \mathcal{D}_p(f). \end{aligned} \tag{2.13.11}$$

□

Theorem 2.13.10 (Sum rule for total derivative). *Let U and V be norm spaces over $\mathbb{R}$, and $f, g : U \to V$ be differentiable at $p \in U$. Then $f + g$ is differentiable at p, and*

$$\mathcal{D}_p(f + g) = \mathcal{D}_p(f) + \mathcal{D}_p(g). \tag{2.13.12}$$

Proof. Let $u \in U \setminus \{0\}$. Then by the sub-additivity of the norm

$$\begin{aligned}
&\frac{\|\Delta_p(f+\beta g)(u) - (\mathcal{D}_p(f)(u) + \mathcal{D}_p(g)(u))\|}{\|u\|} \\
=&\frac{\|[\Delta_p(f)(u) - \mathcal{D}_p(f)(u)] + [\Delta_p(g)(u) - \mathcal{D}_p(g)(u)]\|}{\|u\|} \\
\leq&\frac{\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\|}{\|u\|} + \frac{\|\Delta_p(g)(u) - \mathcal{D}_p(g)(u)\|}{\|u\|}.
\end{aligned} \tag{2.13.13}$$

Now

1. $\lim_{u\to 0} \frac{\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\|}{\|u\|} = 0$, since f is differentiable at p,

2. $\lim_{u\to 0} \frac{\|\Delta_p(g)(u) - \mathcal{D}_p(g)(u)\|}{\|u\|} = 0$, since g is differentiable at p.

Therefore

$$\lim_{u\to 0} \frac{\|\Delta_p(f+g)(u) - (\mathcal{D}_p(f)(u) + \mathcal{D}_p(g)(u))\|}{\|u\|} \leq 0, \tag{2.13.14}$$

by Theorem 2.11.3 and Theorem 2.11.4. The $\mathcal{D}_p(f) + \mathcal{D}_p(g)$ is continuous since V is a topological vector space. The result follows, since total derivative is unique by Theorem 2.13.3. □

Example 2.13.11. Let U and V be norm spaces over $\mathbb{R}$, and $f : U \to V$ be non-differentiable at $p \in U$. Then $\mathcal{D}_p(f - f) = 0$, but $\mathcal{D}_p(f) - \mathcal{D}_p(f)$ does not exist.

Theorem 2.13.12 (Total derivative of a continuous linear function is itself). *Let U and V be norm spaces over $\mathbb{R}$, $f : U \to V$ be continuous linear, and $p \in U$. Then f is differentiable, and $\mathcal{D}_p(f) = f$.*

Proof. Let $u \in U \setminus \{0\}$. Then

$$\begin{aligned}
\|\Delta_p(f)(u) - f(u)\| &= \|f(p+u) - f(p) - f(u)\| \\
&= \|f(p+u) - f(p+u)\| \\
&= 0.
\end{aligned} \tag{2.13.15}$$

□

Theorem 2.13.13 (Product rule for total derivative). *Let U be a norm space, and $f, g : U \to \mathbb{R}$ be differentiable at $p \in U$. Then fg is differentiable at p, and*

$$\mathcal{D}_p(fg) = f(p)\mathcal{D}_p(g) + \mathcal{D}_p(f)g(p). \tag{2.13.16}$$

Proof. First rewrite

$$\begin{aligned}
&\Delta_p(fg)(u) - [\mathcal{D}_p(f)(u)g(p) + f(p)\mathcal{D}_p(g)(u)] \\
=&f(p+u)[\Delta_p(g)(u) - \mathcal{D}_p(g)(u)] \\
+&[g(p) + \mathcal{D}_p(g)(u)][\Delta_p(f)(u) - \mathcal{D}_p(f)(u)] \\
+&\mathcal{D}_p(f)(u)\mathcal{D}_p(g)(u).
\end{aligned} \tag{2.13.17}$$

By the sub-additivity of the norm

$$\begin{aligned}
&\frac{\|\Delta_p(fg)(u) - [\mathcal{D}_p(f)(u)g(p) + f(p)\mathcal{D}_p(g)(u)]\|}{\|u\|} \\
\leq &|f(p+u)|\frac{\|\Delta_p(g)(u) - \mathcal{D}_p(g)(u)\|}{\|u\|} \\
+&|g(p) + \mathcal{D}_p(g)(u)|\frac{\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\|}{\|u\|} \\
+&|\mathcal{D}_p(f)(u)|\left\|\mathcal{D}_p(g)\left(\frac{u}{\|u\|}\right)\right\|.
\end{aligned} \tag{2.13.18}$$

Since f and g are differentiable, they are also continuous by Theorem 2.13.16. Now

- $\lim_{u\to 0} |f(p+u)| = 0$, since norm and f are continuous,
- $\lim_{u\to 0} |g(p) + \mathcal{D}_p(g)(u)| = |g(p)| < \infty$, since norm is continuous, and $\mathcal{D}_p(g)$ is continuous linear,
- $\lim_{u\to 0} |\mathcal{D}_p(f)(u)| = 0$, since norm is continuous, and $\mathcal{D}_p(f)$ is continuous linear,
- $\left\|\mathcal{D}_p(g)\left(\frac{u}{\|u\|}\right)\right\| \leq \|\mathcal{D}_p(g)\| < \infty$, since $\mathcal{D}_p(g)$ is bounded,
- $\lim_{u\to 0} \frac{\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\|}{\|u\|} = 0$, since f is differentiable,
- $\lim_{u\to 0} \frac{\|\Delta_p(g)(u) - \mathcal{D}_p(g)(u)\|}{\|u\|} = 0$, since g is differentiable.

Therefore

$$\frac{\|\Delta_p(fg)(u) - [\mathcal{D}_p(f)(u)g(p) + f(p)\mathcal{D}_p(g)(u)]\|}{\|u\|} \leq 0 \tag{2.13.19}$$

by Theorem 2.11.3 and Theorem 2.11.4. □

Theorem 2.13.14 (Inverse rule for total derivative). *Let U be a norm space, and $f : U \to \mathbb{R}$ be differentiable at $p \in U$, such that $f(p) \neq 0$. Then $1/f$ is differentiable at p, and*

$$\mathcal{D}_p(1/f) = -\frac{\mathcal{D}_p(f)}{f(p)^2}. \tag{2.13.20}$$

Proof. Let $g : U \to \mathbb{R}$ such that $g = 1/f$. Then

$$\begin{aligned}
0 &= \mathcal{D}_p(1) \\
&= \mathcal{D}_p(fg) \\
&= \mathcal{D}_p(f)g(p) + f(p)\mathcal{D}_p(g)
\end{aligned} \tag{2.13.21}$$

by Theorem 2.13.13 and Theorem 2.13.8. Therefore

$$\begin{aligned}
\mathcal{D}_p(g) &= -\frac{\mathcal{D}_p(f)g(p)}{f(p)} \\
&= -\frac{\mathcal{D}_p(f)}{f(p)^2}.
\end{aligned} \tag{2.13.22}$$

□

Theorem 2.13.15 (Quotient rule for total derivative). *Let U be a norm space, and $f, g : U \to \mathbb{R}$ be differentiable at $p \in U$, such that $g(p) \neq 0$. Then f/g is differentiable at p, and*

$$\mathcal{D}_p(f/g) = \frac{\mathcal{D}_p(f)g(p) - f(p)\mathcal{D}_p(g)}{g(p)^2}. \tag{2.13.23}$$

Proof. Now

$$\begin{aligned}
\mathcal{D}_p(f/g) &= \mathcal{D}_p(f(1/g)) \\
&= \frac{\mathcal{D}_p(f)}{g(p)} + f(p)\mathcal{D}_p(1/g) \\
&= \frac{\mathcal{D}_p(f)}{g(p)} - f(p)\frac{\mathcal{D}_p(g)}{g(p)^2} \\
&= \frac{\mathcal{D}_p(f)g(p) - f(p)\mathcal{D}_p(g)}{g(p)^2},
\end{aligned} \tag{2.13.24}$$

where we used Theorem 2.13.14, and Theorem 2.13.13. □

Theorem 2.13.16 (Differentiable is continuous). *Let U and V be norm spaces, and $f : U \to V$ be differentiable at $p \in U$. Then f is continuous at p.*

Proof. Let $u \in U \setminus \{0\}$. Then by the sub-additivity of the norm

$$\begin{aligned}
\|\Delta_p(f)(u)\| &= \|\Delta_p(f)(u) - \mathcal{D}_p(f)(u) + \mathcal{D}_p(f)(u)\| \\
&\leq \|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\| + \|\mathcal{D}_p(f)(u)\| \\
&= \frac{\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\|}{\|u\|}\|u\| + \|\mathcal{D}_p(f)(u)\|.
\end{aligned} \tag{2.13.25}$$

We have that

1. $\lim_{u \to 0} \frac{\|\mathcal{D}_p(f)(u) - \Delta_p(f)(u)\|}{\|u\|} = 0$, since f is differentiable at p,
2. $\lim_{u \to 0}\|u\| = \|\lim_{u \to 0} u\| = 0$, since the norm is continuous, and
3. $\lim_{u \to 0}\|\mathcal{D}_p(f)(u)\| = \|\mathcal{D}_p(f)(\lim_{u \to 0} u)\| = 0$, since the norm and $\mathcal{D}_p(f)$ are continuous, and $\mathcal{D}_p(f)$ is linear.

Taking limits on both sides we have that

$$\lim_{u \to 0}\|\Delta_p(f)(u)\| \leq 0 \tag{2.13.26}$$

by Theorem 2.13.10, Theorem 2.13.9, and Theorem 2.13.13. Therefore f is continuous at p. □

Theorem 2.13.17 (Chain rule for total derivative). *Let U, V, and W be norm spaces, and $p \in U$. Let $g : U \to V$ be differentiable at p, and $f : V \to W$ be differentiable at $g(p)$. Then $f \circ g$ is differentiable at p, and*

$$\mathcal{D}_p(f \circ g) = \mathcal{D}_{g(p)}(f) \circ \mathcal{D}_p(g). \tag{2.13.27}$$

Proof. Let $u \in U \setminus \{0\}$, $q = f(p)$, $v = \Delta_p(f)(u)$, and $w = \mathcal{D}_p(f)(u)$. Then

$$\begin{aligned}
&\left\|\Delta_p(g \circ f)(u) - \mathcal{D}_{f(p)}(g)(\mathcal{D}_p(f)(u))\right\| \\
=&\|\Delta_q(g)(v) - \mathcal{D}_q(g)(w)\| \\
=&\|\Delta_q(g)(v) - \mathcal{D}_q(g)(v) + \mathcal{D}_q(g)(v) - \mathcal{D}_q(g)(w)\| \\
\leq&\|\Delta_q(g)(v) - \mathcal{D}_q(g)(v)\| + \|\mathcal{D}_q(g)(v - w)\| \\
\leq&\|\Delta_q(g)(v) - \mathcal{D}_q(g)(v)\| + \|\mathcal{D}_q(g)\|\|v - w\| \\
=&\|\Delta_q(g)(v) - \mathcal{D}_q(g)(v)\| + \|\mathcal{D}_q(g)\|\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\|
\end{aligned} \tag{2.13.28}$$

Since f is differentiable at p, it is also continuous at p by Theorem 2.13.16. Therefore $\lim_{u \to 0} \Delta_p(f)(u) = 0$. Since f is differentiable, and g is differentiable, for every $\epsilon > 0$ we may choose $\delta > 0$ such that

$$\begin{aligned}
\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\| < \epsilon\|u\|, \\
\|\Delta_q(g)(v) - \mathcal{D}_q(g)(v)\| < \epsilon\|v\|,
\end{aligned} \tag{2.13.29}$$

for all $\|u\| < \delta$. Therefore

$$\begin{aligned}
&\left\|\Delta_p(g \circ f)(u) - \mathcal{D}_{f(p)}(g)(\mathcal{D}_p(f)(u))\right\| \\
\leq&\epsilon\|v\| + \epsilon\|\mathcal{D}_q(g)\|\|u\| \\
=&\epsilon\|v - \mathcal{D}_p(f)(u) + \mathcal{D}_p(f)(u)\| + \epsilon\|\mathcal{D}_q(g)\|\|u\| \\
\leq&\epsilon\|\Delta_p(f)(u) - \mathcal{D}_p(f)(u)\| + \epsilon\|\mathcal{D}_p(f)(u)\| + \epsilon\|\mathcal{D}_q(g)\|\|u\| \\
\leq&\epsilon^2\|u\| + \epsilon\|\mathcal{D}_p(f)\|\|u\| + \epsilon\|\mathcal{D}_q(g)\|\|u\| \\
\leq&\epsilon\|u\|(\epsilon + \|\mathcal{D}_p(f)\| + \|\mathcal{D}_q(g)\|),
\end{aligned} \tag{2.13.30}$$

for all $\|u\| < \delta$. Since this holds for any $\epsilon > 0$, the result holds. □

3 Symmetry groups

3.1 General linear group

Let V be a vector space. The **general linear group** $\mathbf{GL}(V)$ of V is the set of invertible linear functions in V, together with function composition as the group operation. A **linear group** is any sub-group of $\mathbf{GL}(V)$. A linear group S is called **special**, if $\det(f) > 0$ for all $f \in S$.

Theorem 3.1.1 (Composition of linear functions is linear). *Let U, V, and W be vector spaces over F, and $f : U \to V$ and $g : V \to W$ be linear. Then $f \circ g$ is linear.*

Proof. Let $\alpha, \beta \in F$, and $x, y \in U$. Then

$$\begin{aligned}
(f \circ g)(\alpha x + \beta y) &= f(g(\alpha x + \beta y)) \\
&= f(\alpha g(x) + \beta g(y)) \\
&= \alpha f(g(x)) + \beta f(g(y)) \\
&= \alpha(f \circ g)(x) + \beta(f \circ g)(y).
\end{aligned} \tag{3.1.1}$$

□

Theorem 3.1.2 (Inverse of a linear function is linear). *Let V and W be vector spaces over F, and $f : V \to W$ be linear and invertible. Then f^{-1} is linear.*

Proof. Let $\alpha, \beta \in F$, and $x, y \in W$. Then

$$\begin{aligned} f^{-1}(\alpha x + \beta y) &= f^{-1}\big[\alpha f(f^{-1}(x)) + \beta f(f^{-1}(y))\big] \\ &= f^{-1}\big[f(\alpha f^{-1}(x) + \beta f^{-1}(y))\big] \\ &= \alpha f^{-1}(x) + \beta f^{-1}(y). \end{aligned} \tag{3.1.2}$$

□

Theorem 3.1.3 (General linear group is a group). *The general linear group $\mathbf{GL}(V)$ is a group under composition.*

Proof. Composition is associative, and by Theorem 3.1.1 $\mathbf{GL}(V)$ is closed under composition. The identity function is linear and is the identity element under composition. By definition, each element of $\mathbf{GL}(V)$ has an inverse, and by Theorem 3.1.2 $\mathbf{GL}(V)$ is closed under inverses. □

3.2 Scaling linear group

Let V be a vector space over F. A function $f : V \to V$ is called a **scaling**, if there exists $\lambda \in F$ such that

$$f(x) = \lambda x. \tag{3.2.1}$$

The **scaling linear group** is the set of invertible scalings in V, together with function composition as the group operation.

Theorem 3.2.1 (Scaling linear group is a commutative linear group). *The scaling linear group $\mathbf{S}(V)$ is a commutative sub-group of the general linear group $\mathbf{GL}(V)$.*

Proof. Let $f, g \in \mathbf{S}(V)$, such that

$$\begin{aligned} f(x) &= \alpha x, \\ g(x) &= \beta x, \end{aligned} \tag{3.2.2}$$

for some $\alpha, \beta \in F \setminus \{0\}$. Then

$$(f \circ g)(x) = \alpha\beta x. \tag{3.2.3}$$

Since $\alpha\beta \in F \setminus \{0\}$, $f \circ g \in \mathbf{S}(V)$, and $\mathbf{S}(V)$ is closed under composition. Since the composition order does not matter, $\mathbf{S}(V)$ is commutative. If $\beta = \alpha^{-1}$, then $(f \circ g)(x) = x$, and thus $f^{-1} = g$. Since composition is associative, f^{-1} is unique by Theorem 2.7.3. Therefore every element of $\mathbf{S}(V)$ has an inverse. Since $f^{-1} \in \mathbf{S}(V)$, $\mathbf{S}(V)$ is closed under inverses. The identity function is the identity element in $\mathbf{S}(V)$, and is unique by Theorem 2.7.2. Therefore $\mathbf{S}(V)$ is a commutative group under composition. Since every $f \in \mathbf{S}(V)$ is clearly linear, $\mathbf{S}(V)$ is a sub-group of the general linear group $\mathbf{GL}(V)$. □

3.3 Orthogonal linear group

Let V and W be symmetric bilinear spaces over F. The **orthogonal** $\mathbf{O}(V)$ in V is the set of invertible orthogonal linear functions in V. A **plane reflection** in V, defined only when $\text{char}(F) \neq 2$, is a function $\pi_b \in \mathbf{O}(V)$ defined by

$$\pi_b(x) = x - 2\frac{b \cdot x}{b \cdot b}b, \tag{3.3.1}$$

where $b \in V$, $b \cdot b \neq 0$, is the normal of the plane.

Theorem 3.3.1 (Orthogonal is linear when non-degenerate). *Let V and W be bilinear spaces, and $f : V \to W$ be an orthogonal function, such that the bilinear form in W is left-non-degenerate, or right-non-degenerate, on $f(V)$. Then f is linear.*

Proof. We will assume the bilinear form in W is left-non-degenerate on $f(V)$; the right-non-degenerate case is similar. Let $\alpha, \beta \in \mathbb{R}$, and $x, y, z \in V$. Since f is orthogonal,

$$\begin{aligned} f(\alpha x + \beta y) \cdot f(z) &= (\alpha x + \beta y) \cdot z \\ &= \alpha(x \cdot z) + \beta(y \cdot z) \\ &= \alpha(f(x) \cdot f(z)) + \beta(f(y) \cdot f(z)) \\ &= (\alpha f(x) + \beta f(y)) \cdot f(z). \end{aligned} \tag{3.3.2}$$

We write this as

$$(f(\alpha x + \beta y) - (\alpha f(x) + \beta f(y))) \cdot f(z) = 0. \tag{3.3.3}$$

Since $\cdot$ is left-non-degenerate on $f(V)$,

$$f(\alpha x + \beta y) - (\alpha f(x) + \beta f(y)) = 0. \tag{3.3.4}$$

Therefore f is linear. □

Theorem 3.3.2 (Composition of orthogonal functions is orthogonal). *Let U, V, and W be bilinear spaces, and let $f : U \to V$ and $g : V \to W$ be orthogonal. Then $f \circ g$ is orthogonal.*

Proof. Let $x, y \in U$. Then

$$\begin{aligned} (f \circ g)(x) \cdot (f \circ g)(y) &= f(g(x)) \cdot f(g(y)) \\ &= g(x) \cdot g(y) \\ &= x \cdot y. \end{aligned} \tag{3.3.5}$$

□

Theorem 3.3.3 (Inverse of an orthogonal function is orthogonal). *Let V, and W be bilinear spaces, and let $f : V \to W$ be orthogonal and invertible. Then f^{-1} is orthogonal.*

Proof. Let $x, y \in W$. Then

$$\begin{aligned} f^{-1}(x) \cdot f^{-1}(y) &= f(f^{-1}(x)) \cdot f(f^{-1}(y)) \\ &= x \cdot y. \end{aligned} \tag{3.3.6}$$

□

Theorem 3.3.4 (Orthogonal group is a linear group). *The orthogonal group* $\mathbf{O}(V)$ *is a sub-group of the general linear group* $\mathbf{GL}(V)$.

Proof. Composition is associative, and by Theorem 3.3.2 $\mathbf{O}(V)$ is closed under composition. Since the identity function is orthogonal, $\mathbf{O}(V)$ has an identity element, which is unique by Theorem 2.7.3. By definition, every element of $\mathbf{O}(V)$ has an inverse. By Theorem 3.3.3 $\mathbf{O}(V)$ is closed under inverse. Since in addition $\mathbf{O}(V) \subset \mathbf{GL}(V)$, $\mathbf{O}(V)$ is a sub-group of $\mathbf{GL}(V)$. □

Theorem 3.3.5 (Properties of plane reflection). *Let* V *be a bilinear space over* F*, with* $char(F) \neq 2$*. Then the plane reflection* π_b *is linear, orthogonal, invertible, and involutive, for all* $b \in V$, $b \cdot b \neq 0$.

Proof. Let $b \in V$ such that $b \cdot b \neq 0$, and let $\gamma = \frac{1}{b \cdot b}$. Then

$$\pi_b(x) = x - 2\gamma(b \cdot x)b. \tag{3.3.7}$$

Let $x, y \in V$, and $\alpha, \beta \in F$. Then

$$\begin{aligned}\pi_b(\alpha x + \beta y) &= (\alpha x + \beta y) - 2\gamma(b \cdot (\alpha x + \beta y))b \\ &= \alpha[x - 2\gamma(b \cdot x)b] + \beta[y - 2\gamma(b \cdot y)b] \\ &= \alpha\pi_b(x) + \beta\pi_b(y).\end{aligned} \tag{3.3.8}$$

Therefore π_b is linear. Also

$$\begin{aligned}\pi_b(x) \cdot \pi_b(y) &= (x - 2\gamma(b \cdot x)b) \cdot (y - 2\gamma(b \cdot y)b) \\ &= x \cdot y - 4\gamma(b \cdot x)(b \cdot y) + 4\gamma^2(b \cdot x)(b \cdot y)(b \cdot b) \\ &= x \cdot y - 4\gamma(b \cdot x)(b \cdot y) + 4\gamma(b \cdot x)(b \cdot y) \\ &= x \cdot y.\end{aligned} \tag{3.3.9}$$

Therefore π_b is orthogonal. Now

$$\begin{aligned}(\pi_b \circ \pi_b)(x) &= \pi_b(x) - 2\gamma(b \cdot \pi_b(x))b \\ &= (x - 2\gamma(b \cdot x)b) - 2\gamma(b \cdot (x - 2\gamma(b \cdot x)b))b \\ &= x - 4\gamma(b \cdot x)b + 4\gamma^2(b \cdot x)(b \cdot b)b \\ &= x - 4\gamma(b \cdot x)b + 4\gamma(b \cdot x)b \\ &= x.\end{aligned} \tag{3.3.10}$$

Therefore π_b is invertible and involutive. □

Theorem 3.3.6 (Orthogonal function by reflections). *Let* V *be an* n*-dimensional non-degenerate symmetric bilinear space over* F*, with* $n \geq 1$ *and* $char(F) \neq 2$*, and* $f \in \mathbf{O}(V)$*. Then there exists a set of functions* $\pi_1, \dots, \pi_n : V \to V$*, each either a plane reflection or an identity function, such that*

$$\pi_1 \circ \cdots \circ \pi_n = f. \tag{3.3.11}$$

Proof. Consider the subspace

$$B = \{f(a) - a : a \in V\}. \tag{3.3.12}$$

Suppose every vector in B is null. TODO.

Otherwise, there exists $a \in V$ such that $b \cdot b \neq 0$, where $b = f(a) - a$. Then

$$\begin{aligned} (b \cdot b)\pi_b(a) &= (b \cdot b)a - 2(b \cdot a)b \\ &= (b \cdot b)f(a) - (b \cdot b)b - 2(b \cdot a)b \\ &= (b \cdot b)f(a) - (b \cdot (f(a) + a))b \\ &= (b \cdot b)f(a), \end{aligned} \tag{3.3.13}$$

where the last step follows from the orthogonality of f by

$$\begin{aligned} b \cdot (f(a) + a) &= (f(a) - a) \cdot (f(a) + a) \\ &= f(a) \cdot f(a) - a \cdot a \\ &= 0. \end{aligned} \tag{3.3.14}$$

Since π_b maps $a \mapsto f(a)$, we may choose $\pi_1 = \pi_b$, and the claim holds for $n = 1$. Assume the claim holds for $n - 1$, where $n > 1$. Let $S = \operatorname{span}(f(a)) \subset V$, and $\widehat{f} \in \mathbf{O}\big(S^{\Vdash V}\big)$ such that $\widehat{f}$ is the restriction of $\pi_1 \circ f$ to $S^{\Vdash V}$. Since $S^{\Vdash V}$ is $(n-1)$-dimensional, there exists functions $\widehat{\pi}_2, \dots \widehat{\pi}_n : S^{\Vdash V} \to S^{\Vdash V}$ of the required type such that

$$\widehat{f} = \widehat{\pi}_2 \circ \cdots \circ \widehat{\pi}_n. \tag{3.3.15}$$

We extend these functions to linear functions $\pi_2, \dots, \pi_n : V \to V$ such that

$$\pi_i(x) = \begin{cases} x, & \text{if } x \in S, \\ \widehat{\pi}_i(x), & \text{if } x \in S^{\Vdash V}. \end{cases} \tag{3.3.16}$$

for all $2 \leq i \leq n$. This extension by identity retains all the required properties. Then

$$\pi_2 \circ \cdots \circ \pi_n = \pi_1 \circ f. \tag{3.3.17}$$

Since π_1 is involutive,

$$\pi_1 \circ \cdots \circ \pi_n = f. \tag{3.3.18}$$

$\square$

Remark 3.3.7. Theorem 3.3.6 is known as the **Cartan-Dieudonné theorem**. It states that the linear plane reflections generate the orthogonal linear group. The usefulness of this theorem lies in that if we can represent plane reflections and their compositions in some algebra, then we can also represent any orthogonal function in that algebra. This is particularly true for geometric algebra. It is this theorem which highlights the importance of having a bilinear form which is both symmetric and non-degenerate.

Remark 3.3.8 (Orientations of subspaces). Consider the decomposition $V = V^- \perp V^+$. For a transform $f \in \mathbf{O}(V)$ it must hold that $f(V^-) = V^-$, and $f(V^+) = V^+$. Therefore we may decompose $f = f^- \oplus f^+$, where $f^- \in \mathbf{O}(V^-)$, and $f^+ \in \mathbf{O}(V^+)$. Then $\det(f) = \det(f^-)\det(f^+)$, and there are four cases, corresponding to the four connected components of $\mathbf{O}(V)$. These cases are shown in Table 7.

Table 7: Types of transformations in $\mathbf{O}(V)$ by orientation change. Here $f = f^+ \oplus f^- \in \mathbf{O}(V)$, where $f^+ \in \mathbf{O}(V^+)$, and $f^- \in \mathbf{O}(V^-)$.

$\det(f^+)$	$\det(f^-)$	$\det(f)$	Component
-1	-1	$+1$	$\mathbf{SO}^-(V)$
-1	$+1$	-1	$\mathbf{O}^-(V)$
$+1$	-1	-1	$\mathbf{O}^-(V)$
$+1$	$+1$	$+1$	$\mathbf{SO}^+(V)$

3.4 Conformal linear group

Let V be a symmetric bilinear space over F. The **conformal linear group** $\mathcal{CO}(V)$ is the direct sum

$$\mathcal{CO}(V) = \mathbf{S}(V) \oplus \mathbf{O}(V). \tag{3.4.1}$$

Remark 3.4.1. Conformal means angle-preserving.

Remark 3.4.2. $\mathcal{CO}(V)$ is also known as the conformal orthogonal group. We find this terminology confusing, since a conformal linear transformation need not be orthogonal.

Theorem 3.4.3 (Characterization of conformal linear functions). *Let $f \in \mathcal{CO}(V)$. Then there exists $\lambda_f \in F$, such that*

$$f(x) \cdot f(y) = \lambda_f (x \cdot y), \tag{3.4.2}$$

for all $x, y \in V$.

Proof. Scalings commute with orthogonal functions. Therefore one can rearrange any composition of functions such that scalings are done first, followed by orthogonal functions. Since the composition of scalings is a scaling, and the composition of orthogonal functions is orthogonal, the result follows. □

3.5 Affine groups

Let V and W be a vector spaces over F. A function $f : V \to W$ is called **affine**, if

$$f((1-\alpha)x + \alpha y) = (1-\alpha)f(x) + \alpha f(y), \tag{3.5.1}$$

for all $\alpha \in \mathbb{R}$, and $x, y \in V$. A function $f : V \to V$ is called a **translation**, if there exists $t \in V$ such that

$$f(x) = x + t. \tag{3.5.2}$$

The **translation affine group** $\mathbf{T}(V)$ is the set of translations in V, together with function composition as the group operation. An **affine group** is a direct sum of the translation affine group and any subgroup of the general linear group.

Example 3.5.1. Translation is affine. Except for the identity translation, it is not linear.

Example 3.5.2. Examples of affine groups include general affine group, translation affine group, scaling affine group, orthogonal affine group, and conformal affine group.

Theorem 3.5.3 (An affine group is a commutative group). *The translation affine group* $\mathbf{T}(V)$ *is a commutative group under composition.*

Proof. Obvious. $\square$

Theorem 3.5.4 (Affine generalizes linear). *Any linear group is a sub-group of its corresponding affine group.*

Proof. Obvious. $\square$

4 Algebraic structure

4.1 Clifford algebra

Let V be a symmetric bilinear space over $\mathbb{R}$, and $W \subset T(V)$ a subspace of $T(V)$ defined by

$$W = \operatorname{span}\big(\{A_k v^2 B_l - (v \cdot v) A_k B_l : k, l \in \mathbb{N}, v \in V, A_k \in V^{\otimes k}, B_l \in V^{\otimes l}\}\big). \tag{4.1.1}$$

Then the **Clifford algebra** $\mathcal{C}l(V)$ on V is the quotient algebra

$$\mathcal{C}l(V) = T(V)/W. \tag{4.1.2}$$

The symmetric bilinear form in V is called the **dot product**. If we want to be explicit about the signature (p, q, r) of the dot product, then we denote $\mathcal{C}l(V) = \mathcal{C}l(V)_{p,q,r}$, or $\mathcal{C}l(V) = \mathcal{C}l(V)_{p,q}$, if $r = 0$. The product in $\mathcal{C}l(V)$ is called the **geometric product**. The elements of $\mathcal{C}l(V)$ are called **multi-vectors**. If $A \in \mathcal{C}l(V)$, and $A = a_1 \cdots a_k$ for some $\{a_1, \dots, a_k\} \subset V$, then A is called a k**-versor**. An element $A_k \in \mathcal{C}l(V)$ is called a k**-blade** if it is a k-versor of orthogonal vectors, and a k**-vector**, if it is a linear combination of k-blades. The **center** of $\mathcal{C}l(V)$ is the sub-algebra

$$\operatorname{Center}(\mathcal{C}l(V)) = \{A \in \mathcal{C}l(V) : AB = BA, \text{ for all } B \in \mathcal{C}l(V)\}. \tag{4.1.3}$$

Remark 4.1.1. The *Cl* in $\mathcal{C}l(V)$ stands for William Kingdon Clifford, the inventor of Clifford algebra.

Remark 4.1.2 (Notation for Clifford algebra). Taking the notation for the tensor algebra as a starting point, and writing the equivalence classes of $\mathcal{C}l(V)$ by their representatives, we may write equations such as

$$(2uw - 4v)w = 2uw^2 - 4vw, \tag{4.1.4}$$

for $u, v, w \in V$. We interpret the quotient $T(V)/W$ as stating a simplification rule in this notation, so that it makes sense to further write

$$(2uw - 4v)w = 2(w \cdot w)u - 4vw. \tag{4.1.5}$$

Remark 4.1.3 (Naming for versors). Some texts, such as [3], require a versor to be invertible. However, many theorems apply to general products of vectors rather than just to invertible products of vectors. We follow [4] in our naming.

Remark 4.1.4 ($\mathbb{R}$ is a sub-algebra of $\mathcal{C}l(V)$). The sub-algebra $\{\alpha e_0 : \alpha \in \mathbb{R}\} \subset \mathcal{C}l(V)$ is field-isomorphic to $\mathbb{R}$; the isomorphism is $\phi : \mathbb{R} \to \mathcal{C}l(V) : \phi(\alpha) = \alpha e_0$. Therefore these two sets are often interchanged in notation, although a rigorous, but perhaps distracting, way would be to explicitly use e_0 everywhere. Let us temporarily denote the geometric product by $\circledast$. Since for $\alpha, \beta \in \mathbb{R}$ and $A \in \mathcal{C}l(V)$,

$$\begin{aligned}
(\alpha e_0 + \beta e_0) \circledast A &= \alpha(e_0 \circledast A) + \beta(e_0 \circledast A) && (4.1.6)\\
&= (\alpha + \beta)A && (4.1.7)\\
&= \alpha(A \circledast e_0) + \beta(A \circledast e_0) && (4.1.8)\\
&= A \circledast (\alpha e_0 + \beta e_0), && (4.1.9)
\end{aligned}$$

there is no danger in this abuse of notation.

Theorem 4.1.5 (Dot product from geometric product). *Let $a, b \in V$. Then $\frac{1}{2}(ab + ba) = a \cdot b$.*

Proof.

$$\begin{aligned}
ab + ba &= (a + b)^2 - a^2 - b^2 && (4.1.10)\\
&= (a + b) \cdot (a + b) - a \cdot a - b \cdot b && (4.1.11)\\
&= 2a \cdot b. && (4.1.12)
\end{aligned}$$

□

Theorem 4.1.6 (Orthogonal vectors anti-commute). *If $a, b \in V$, then*

$$a \cdot b = 0 \Leftrightarrow ab = -ba, \qquad (4.1.13)$$

i.e. vectors are orthogonal if and only if they anti-commute.

Proof. This is an immediate consequence of Theorem 4.1.5. □

Remark 4.1.7 (Coordinate-free proofs). Following [4], we do not pick a priviledged basis for V to define a Clifford algebra $\mathcal{C}l(V)$. This approach reveals more structure in proofs, since the number of assumptions is reduced. This is to be contrasted with some texts picking a priviledged orthogonal basis $\{e_1, \dots, e_n\} \subset V$, and then relying on coordinate expansions and the fact that $e_i e_j = -e_j e_i$, for $i \neq j$ by Theorem 4.1.6. Picking a priviledged orthogonal basis is also problematic when non-orthogonal bases of V are used, as is done with the conformal geometric algebra model. The non-orthogonal basis vectors have then to be described in the terms of the orthogonal basis vectors. However, it is then not immediately clear which results proved for the orthogonal basis vectors hold, or do not hold, for the non-orthogonal basis vectors.

Remark 4.1.8. If $V = \{0\}$, a zero-dimensional vector-space, then $\mathcal{C}l(V)$ is field-isomorphic to $\mathbb{R}$.

4.2 Morphisms

The **reversion** is an algebra anti-automorphism $\widetilde{\ } : \mathcal{C}l(V) \to \mathcal{C}l(V)$ such that $\widetilde{v} = v$ for all $v \in V$. The **grade involution** is an algebra automorphism $\widehat{\ } : \mathcal{C}l(V) \to \mathcal{C}l(V)$ such that $\widehat{v} = -v$ for all $v \in V$. The **conjugation** is an algebra anti-automorphism $\bar{\ } : \mathcal{C}l(V) \to \mathcal{C}l(V)$ such that $\bar{v} = -v$ for all $v \in V$.

Remark 4.2.1. The missing automorphism to fill the pattern is $\mathcal{C}l(V) \to \mathcal{C}l(V)$ such that $v \mapsto v$ for all $v \in V$. But this is just the identity function on $\mathcal{C}l(V)$.

Remark 4.2.2. The reversion, grade involution, and conjugation are all involutions, i.e. they are their own inverses.

Remark 4.2.3. If $\{a_1, \dots, a_k\} \subset V$, then $\widetilde{a_1 \cdots a_k} = a_k \cdots a_1$, explaining the name reversion.

Theorem 4.2.4 (Vector-preserving homomorphisms preserve dot product). *Let $f : \mathcal{C}l(V) \to \mathcal{C}l(W)$ be an algebra homomorphism (anti-homomorphism) such that $f(V) \subset W$. Then $f|V$ is orthogonal.*

Proof. We will prove the result assuming f is an homomorphism; the proof for the anti-homomorphism is almost identical. Let $a, b \in V$. Then by Theorem 4.1.5, and $f(V) \subset W$,

$$\begin{aligned} 2(f(a) \cdot f(b)) &= f(a)f(b) + f(b)f(a) \\ &= f(ab) + f(ba) \\ &= f(ab + ba) \\ &= f(2(a \cdot b)) \\ &= 2(a \cdot b). \end{aligned} \tag{4.2.1}$$

Therefore $f|V$ is orthogonal. □

Theorem 4.2.5 (Reversion formula). *Let $A_k \in \mathcal{C}l(V)$ be a k-vector. Then $\widetilde{A_k} = (-1)^{\frac{k(k-1)}{2}} A_k$.*

Proof. By linearity, we only need to prove the result for k-blades. Let $A_k = a_1 \cdots a_k$, where $\{a_1, \dots, a_k\} \subset V$ is an orthogonal set of vectors. By Theorem 4.1.6, $a_i a_j = -a_j a_i$, for $i \neq j$. Therefore

$$\begin{aligned} \widetilde{A_k} &= \widetilde{a_1 \cdots a_k} \\ &= a_k \cdots a_1 \\ &= (-1)^{k-1} a_1 a_k \cdots a_2 \\ &= (-1)^{k-1} \cdots (-1)^1 a_1 \cdots a_k \\ &= (-1)^{\frac{k(k-1)}{2}} A_k. \end{aligned} \tag{4.2.2}$$

□

Theorem 4.2.6 (Grade involution formula). *Let $A = a_1 \cdots a_k \in \mathcal{C}l(V)$ be a k-versor, where $\{a_1, \dots, a_k\} \subset V$. Then*

$$\widehat{A} = (-1)^k A. \tag{4.2.3}$$

Proof.

$$
\begin{aligned}
\widehat{A} &= \widehat{a_1 \cdots a_k} \\
&= \widehat{a_1} \cdots \widehat{a_k} \\
&= (-a_1) \cdots (-a_k) \\
&= (-1)^k a_1 \cdots a_k \\
&= (-1)^k A.
\end{aligned} \tag{4.2.4}
$$

$\square$

Remark 4.2.7. Theorem 4.2.6 extends by linearity to linear combinations of k-versors, and in particular to k-vectors.

Theorem 4.2.8 (Conjugation formula). *Let $A_k \in \mathcal{C}l(V)$ be a k-vector. Then* $\overline{A_k} = \widetilde{(\widehat{A_k})} = \widehat{(\widetilde{A_k})} = (-1)^{\frac{k(k+1)}{2}} A_k$.

Proof. By linearity, we only need to prove the result for the k-blades. Now

$$
\begin{aligned}
\overline{A_k} &= \overline{a_1 \cdots a_k} \\
&= \overline{a_k} \cdots \overline{a_1} \\
&= (-a_k) \cdots (-a_1) \\
&= (-1)^k a_k \cdots a_1 \\
&= (-1)^k \widetilde{A_k} \\
&= \widehat{(\widetilde{A_k})} \\
&= \widetilde{(\widehat{A_k})} \\
&= (-1)^k (-1)^{\frac{k(k-1)}{2}} A_k \\
&= (-1)^{\frac{k(k+1)}{2}} A_k.
\end{aligned} \tag{4.2.5}
$$

$\square$

4.3 Inverse

Theorem 4.3.1 (Product of a k-versor and its reverse or conjugate is real). *Let $A \in \mathcal{C}l(V)$ be a k-versor. Then*

$$
\begin{aligned}
A\widetilde{A} &= \widetilde{A}A \in \mathbb{R}, \\
A\overline{A} &= \overline{A}A \in \mathbb{R}.
\end{aligned} \tag{4.3.1}
$$

Proof. Let $A = a_1 \cdots a_k \in \mathcal{C}l(V)$, where $\{a_1, \ldots, a_k\} \subset V$. Then

$$\begin{aligned}
A\widetilde{A} &= (a_1 \cdots a_k)(a_k \cdots a_1) \\
&= \prod_{i=1}^{n} a_i^2 \\
&= \prod_{i=1}^{n} a_i \cdot a_i \in \mathbb{R} \\
&= (a_k \cdots a_1)(a_1 \cdots a_k) \\
&= \widetilde{A}A.
\end{aligned} \tag{4.3.2}$$

Similarly, but using the conjugation, one shows that $A\overline{A} = \overline{A}A \in \mathbb{R}$. □

Theorem 4.3.2 (Versor inverse). *Let $A \in \mathcal{C}l(V)$ be a k-versor such that $A\widetilde{A} \neq 0$. Then A has a unique inverse with respect to the geometric product given by*

$$A^{-1} = \frac{\widetilde{A}}{A\widetilde{A}} = \frac{\overline{A}}{A\overline{A}}. \tag{4.3.3}$$

Proof. By Theorem 4.3.1, $A\widetilde{A} = \widetilde{A}A \in \mathbb{R}$. Now

$$A\frac{\widetilde{A}}{A\widetilde{A}} = \frac{A\widetilde{A}}{A\widetilde{A}} = 1 = \frac{\widetilde{A}A}{A\widetilde{A}} = \frac{\widetilde{A}}{A\widetilde{A}}A. \tag{4.3.4}$$

The inverse is unique by Theorem 2.7.3. Similarly, but using the conjugation, one shows the latter equation. □

Theorem 4.3.3 (Blade squared is real). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade. Then*

$$A_k^2 \in \mathbb{R}. \tag{4.3.5}$$

Proof. By Theorem 4.3.1 and Theorem 4.2.5,

$$A_k^2 = (-1)^{\frac{k(k-1)}{2}} A_k\widetilde{A_k} \in \mathbb{R}. \tag{4.3.6}$$

□

Example 4.3.4. While for a k-blade $A_k \in \mathcal{C}l(V)$ it holds that $A_k^2 \in \mathbb{R}$, this does not hold for k-versors in general. As an example, consider $a, b \in \mathcal{C}l(\mathbb{R}^{2,0})$, where $a = e_1$ and $b = e_1 + e_2$. Then $ab = 1 + e_1e_2$, and $(ab)^2 = 2e_1e_2 \notin \mathbb{R}$. This shows that the initially attractive definition A_k/A_k^2 for the inverse of a k-blade does not generalize to k-versors.

4.4 Grading

The **grade** of a k-vector is k. Let $X \in \mathcal{C}l(V)$ such that

$$X = \sum_{k=0}^{n} X_k,$$

where $X_k \in \mathcal{C}l(V)$ is a k-vector. Then the **grade-selection operator** $\langle\cdot\rangle_k : \mathcal{C}l(V) \to \mathcal{C}l(V)$ is defined by $\langle X\rangle_k = X_k$. Let

$$\begin{aligned}\langle X\rangle_+ &= \sum_{k=0}^{\infty}\langle X\rangle_{2k}\\ \langle X\rangle_- &= \sum_{k=0}^{\infty}\langle X\rangle_{2k+1}.\end{aligned} \tag{4.4.1}$$

The **even part** of $\mathcal{C}l(V)$ is the set $\langle\mathcal{C}l(V)\rangle_+$, and the **odd part** of $\mathcal{C}l(V)$ is the set $\langle\mathcal{C}l(V)\rangle_-$. The multi-vectors in these sets are called **even** and **odd**, respectively.

Remark 4.4.1. Although all elements of $\mathcal{C}l(V)$ are vectors in the vector-space sense, in Clifford algebra the vector term is reserved for the elements of V, i.e. 1-vectors.

Remark 4.4.2. The grade selection operator is linear and idempotent. It is a projection to the k-vector subspace.

Theorem 4.4.3 (Grade selection commutes with vector-preserving homomorphisms). *Let $f : \mathcal{C}l(V) \to \mathcal{C}l(W)$ be an algebra homomorphism (anti-homomorphism). Then*

$$f(\langle A\rangle_k) = \langle f(A)\rangle_k \tag{4.4.2}$$

if and only if $f(V) \subset W$.

Proof. We will prove the result assuming f is an algebra homomorphism; the proof for the algebra anti-homomorphism is almost identical. Assume $f(V) \subset W$. By linearity we only need prove the result for l-blades. Let $A_l = a_1 \cdots a_l \in \mathcal{C}l(V)$ be an l-blade, where $\{a_1, \dots, a_l\} \subset V$ is orthogonal. Then

$$\begin{aligned}f(A_l) &= f(a_1 \cdots a_l)\\ &= f(a_1) \cdots f(a_l).\end{aligned} \tag{4.4.3}$$

Then $f(A_l)$ is also an l-blade, since each $f(a_i) \in W$, and by Theorem 4.2.4 $f_{|V}$ is orthogonal. Assume $l \neq k$. Then $f(\langle A_l\rangle_k) = 0 = \langle f(A_l)\rangle_k$, and the result holds. Assume $l = k$. Then

$$\begin{aligned}f(\langle A_k\rangle_k) &= f(A_k)\\ &= \langle f(A_k)\rangle_k,\end{aligned} \tag{4.4.4}$$

and the result holds. Assume $f(\langle A\rangle_k) = \langle f(A)\rangle_k$ holds. Then in particular

$$f(a) = \langle f(a)\rangle_1, \tag{4.4.5}$$

for all $a \in V$. Therefore $f(V) \subset W$. ☐

Remark 4.4.4. In particular, Theorem 4.4.3 applies to reversion, grade involution, and conjugation.

Theorem 4.4.5 (Clifford algebra is Z_2-graded). *Clifford algebra $\mathcal{C}l(V)$ is Z_2-graded.*

Proof. Let $A, B \in \mathcal{C}l(V)$. By Theorem 4.4.3,

$$\begin{aligned}
\left\langle \langle A\rangle_+ \langle B\rangle_+ \right\rangle_k &= \left\langle \widehat{\widehat{\langle A\rangle_+ \langle B\rangle_+}} \right\rangle_k \\
&= (-1)^k \left\langle \widehat{\langle A\rangle_+ \langle B\rangle_+} \right\rangle_k \\
&= (-1)^k \left\langle \widehat{\langle A\rangle_+}\, \widehat{\langle B\rangle_+} \right\rangle_k \\
&= (-1)^k \left\langle \langle A\rangle_+ \langle B\rangle_+ \right\rangle_k .
\end{aligned} \tag{4.4.6}$$

It follows that if k is odd, then $\left\langle \langle A\rangle_+ \langle B\rangle_+ \right\rangle_k = 0$. Repeating this reasoning for the other combinations odd-even, even-odd, and odd-odd, we get that

$$\begin{aligned}
\langle A\rangle_+ \langle B\rangle_+ &\in \langle \mathcal{C}l(V)\rangle_+ \\
\langle A\rangle_- \langle B\rangle_+ &\in \langle \mathcal{C}l(V)\rangle_- \\
\langle A\rangle_+ \langle B\rangle_- &\in \langle \mathcal{C}l(V)\rangle_- \\
\langle A\rangle_- \langle B\rangle_- &\in \langle \mathcal{C}l(V)\rangle_+ .
\end{aligned} \tag{4.4.7}$$

Therefore $\mathcal{C}l(V) = \langle \mathcal{C}l(V)\rangle_+ \dotplus \langle \mathcal{C}l(V)\rangle_-$ is Z_2-graded (think of $+$ as zero, and $-$ as one). ☐

Remark 4.4.6. A Z_2-graded algebra is also called a **super-algebra**.

Remark 4.4.7. The proof of Theorem 4.4.5 also shows that $\langle \mathcal{C}l(V)\rangle_+$ is a sub-algebra of $\mathcal{C}l(V)$, called the **even** sub-algebra of $\mathcal{C}l(V)$, and that $\langle \mathcal{C}l(V)\rangle_-$ is *not* a sub-algebra of $\mathcal{C}l(V)$, since it is not closed under multiplication.

Example 4.4.8 (Clifford algebra is not $\mathbb{N}$-graded). Although we have the decomposition $\mathcal{C}l(V) = \bigoplus_{k=0}^{n} \langle \mathcal{C}l(V)\rangle_k$, and we call the k in k-vector a grade, the Clifford algebra $\mathcal{C}l(V)$ is not $\mathbb{N}$-graded. For example, if $a \in V$, then a^2 has grade 0 although a has grade 1. Instead, as we will see later, it is the exterior algebra of $\mathcal{C}l(V)$ which is $\mathbb{N}$-graded.

Theorem 4.4.9 (The parity of a k-versor equals the parity of k). *Let $A = a_1 \cdots a_k \in \mathcal{C}l(V)$ be a k-versor, where $\{a_1, \cdots, a_k\} \subset V$. Then A is even if and only if k is even, and A is odd if and only if k is odd.*

Proof. If $k = 0$, then the result holds. Assume $k > 0$, and k is even. Since each a_i is odd, by Theorem 4.4.5 $a_i a_{i+1}$ is even, for $i < n$. Then

$$a_1 \cdots a_k = \prod_{i=1}^{k/2} a_{2i-1} a_{2i} \in \langle \mathcal{C}l(V)\rangle_+ . \tag{4.4.8}$$

Assume $k > 0$, and k is odd. Then $a_1 \cdots a_{k-1}$ is even by the previous, and $a_1 \cdots a_{k-1} a_k$ is odd by Theorem 4.4.5. ☐

Theorem 4.4.10 (Grades in a geometric product of a vector and a blade). *Let $a \in V$, and $B_k \in \mathcal{C}l(V)$ be a k-vector. Then*

$$\begin{aligned}
aB_k &= \langle aB_k\rangle_{k-1} + \langle aB_k\rangle_{k+1} \\
B_k a &= \langle B_k a\rangle_{k-1} + \langle B_k a\rangle_{k+1}
\end{aligned} \tag{4.4.9}$$

Proof. By linearity we only need to prove the result for k-blades. Let $B_k = b_1 \cdots b_k$, where $\{b_1, \ldots, b_k\} \subset V$ is an orthogonal set of vectors. Then

$$\begin{aligned}
aB_k &= aB_k \\
&= (a_\| + a_\perp)B_k \\
&= a_\| B_k + a_\perp B_k \\
&= \left(\sum_{i=1}^{k} \alpha_i b_i\right) B_k + a_\perp B_k \\
&= \left(\sum_{i=1}^{k} \alpha_i (-1)^{i-1} (b_i \cdot b_i) b_1 \cdots \check{b}_i \cdots b_k\right) + a_\perp b_1 \cdots b_k \\
&= \langle aB_k \rangle_{k-1} + \langle aB_k \rangle_{k+1},
\end{aligned} \tag{4.4.10}$$

where $a_\| \in \operatorname{span}(b_1, \ldots, b_k)$, $a_\perp \in \operatorname{span}(b_1, \ldots, b_k)^\perp$, and the check-mark denotes a missing factor. Similarly for $B_k a$. $\square$

Theorem 4.4.11 (Grades in a geometric product of blades). *Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-vector, and an l-vector, respectively. Then*

$$A_k B_l = \sum_{i=0}^{m} \langle A_k B_l \rangle_{|k-l|+2i}, \tag{4.4.11}$$

where $m = \frac{1}{2}(k + l - |k - l|)$.

Theorem 4.4.12 (Cancellation of grades). *Let $a \in V$, and $B_k \in \mathcal{C}l(V)$ be a k-vector. Then*

$$\begin{aligned}
aB_k + \widehat{B_k} a &= \left\langle aB_k + \widehat{B_k} a \right\rangle_{k-1} \\
aB_k - \widehat{B_k} a &= \left\langle aB_k - \widehat{B_k} a \right\rangle_{k+1}.
\end{aligned} \tag{4.4.12}$$

Proof. By Theorem 4.4.3,

$$\begin{aligned}
\langle aB_k \rangle_{k-1} &= \widetilde{\widetilde{\langle aB_k \rangle}}_{k-1} \\
&= \left\langle \widetilde{\widetilde{aB_k}} \right\rangle_{k-1} \\
&= (-1)^{\frac{(k-1)(k-2)}{2}} \left\langle \widetilde{B_k} \widetilde{a} \right\rangle_{k-1} \\
&= (-1)^{\frac{(k-1)(k-2)}{2}} (-1)^{\frac{k(k-1)}{2}} \langle B_k a \rangle_{k-1} \\
&= (-1)^{(k-1)^2} \langle B_k a \rangle_{k-1} \\
&= (-1)^{k-1} \langle B_k a \rangle_{k-1} \\
&= -\left\langle \widehat{B_k} a \right\rangle_{k-1},
\end{aligned} \tag{4.4.13}$$

where we used the identity $(-1)^{(k-1)^2} = (-1)^{k-1}$. By Theorem 4.4.11,

$$\begin{aligned} aB_k + \widehat{B_k}a &= \left\langle aB_k + \widehat{B_k}a \right\rangle_{k-1} + \left\langle aB_k + \widehat{B_k}a \right\rangle_{k+1} \\ &= \left\langle aB_k + \widehat{B_k}a \right\rangle_{k+1}. \end{aligned} \tag{4.4.14}$$

Similarly for $aB_k - \widehat{B_k}a$. □

Theorem 4.4.13 (Factors can be swapped in 0-grade-selection). *Let $A, B \in \mathcal{C}l(V)$. Then*

$$\langle AB \rangle_0 = \langle BA \rangle_0. \tag{4.4.15}$$

Proof. By linearity, we only need to prove the result for k-vectors. Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-vector, and an l-vector, respectively. Assume $k \neq l$. Then by Theorem 4.4.11 $\langle A_k B_l \rangle_0 = 0 = \langle B_l A_k \rangle$, and the result holds. Assume $k = l$. Then by Theorem 4.4.3

$$\begin{aligned} \langle A_k B_k \rangle_0 &= \widetilde{\langle A_k B_k \rangle}_0 \\ &= \left\langle \widetilde{A_k B_k} \right\rangle_0 \\ &= \left\langle \widetilde{B_k} \widetilde{A_k} \right\rangle_0 \\ &= \langle B_k A_k \rangle_0, \end{aligned} \tag{4.4.16}$$

and the result holds. □

4.5 Exterior product

The **exterior product** is the function $\wedge : \mathcal{C}l(V)^2 \to \mathcal{C}l(V)$ defined by

$$A \wedge B = \sum_{k=0}^{n} \sum_{l=0}^{n} \langle A_k B_l \rangle_{k+l}. \tag{4.5.1}$$

The **exterior algebra** of V is the vector space $\mathcal{C}l(V)$ equipped with the exterior product, denoted by $G(V)$. An **outer-morphism** from V to W is an algebra homomorphism $f : G(V) \to G(W)$ such that $f(V) \subset W$. The set of outer-morphisms from V to W is denoted by $\mathrm{Out}(V, W)$. We also denote $\mathrm{Out}(V) = \mathrm{Out}(V, V)$.

Remark 4.5.1. The G in $G(V)$ stands for Hermann Grassmann, the inventor of exterior algebra.

Theorem 4.5.2 (Exterior product is bilinear). *The exterior product is bilinear.*

Proof. The geometric product is bilinear, and the grade selection operator is linear. □

Theorem 4.5.3 (Exterior product is associative). *The exterior product is associative.*

Proof. By bilinearity, we only need to show associativity for k-vectors. Let $A_k, B_l, C_m \in \mathcal{C}l(V)$, where A_k is a k-vector, B_l is an l-vector, and C_m is an m-vector. Then

$$\begin{aligned}
(A_k \wedge B_l) \wedge C_m &= \langle A_k B_l \rangle_{k+l} \wedge C_m && (4.5.2)\\
&= \left\langle \langle A_k B_l \rangle_{k+l} C_m \right\rangle_{(k+l)+m} && (4.5.3)\\
&= \left\langle \langle A_k B_l C_m \rangle_{(k+l)+m} \right\rangle_{(k+l)+m} && (4.5.4)\\
&= \left\langle \langle A_k B_l C_m \rangle_{k+(l+m)} \right\rangle_{k+(l+m)} && (4.5.5)\\
&= \left\langle A_k \langle B_l C_m \rangle_{l+m} \right\rangle_{k+(l+m)} && (4.5.6)\\
&= A_k \wedge \langle B_l C_m \rangle_{l+m} && (4.5.7)\\
&= A_k \wedge (B_l \wedge C_m). && (4.5.8)
\end{aligned}$$

□

Theorem 4.5.4 (Exterior product with a scalar). *If $B \in \mathcal{C}l(V)$, and $\alpha \in \mathbb{R}$, then $\alpha \wedge B = \alpha B = B \wedge \alpha$.*

Proof. By bilinearity, we only need to prove the result for the k-vectors. Let $\alpha \in \mathbb{R}$, and $B_k \in \mathcal{C}l(V)$ be a k-vector. Then

$$\alpha \wedge B_k = \langle \alpha B_k \rangle_k = \alpha \langle B_k \rangle_k = \alpha B_k.$$

Similarly, $B_k \wedge \alpha = \alpha B_k$. □

Theorem 4.5.5 (Exterior product is alternating for blades). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade. Then*

$$A_k \wedge A_k = 0. \qquad (4.5.9)$$

Proof. Since by Theorem 4.3.3 $A_k^2 \in \mathbb{R}$,

$$\begin{aligned}
A_k \wedge A_k &= \left\langle A_k^2 \right\rangle_2 \\
&= 0.
\end{aligned} \qquad (4.5.10)$$

□

Example 4.5.6 (Exterior product is not alternating in general). Theorem 4.5.5 does not hold in general for multi-vectors; for example, $1 \wedge 1 = 1 \neq 0$. It does not hold in general even for k-vectors; for example, if $\{e_1, e_2, e_3, e_4\} \subset V$ is linearly independent, then by Theorem 4.5.7

$$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4) = 2(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \neq 0. \qquad (4.5.11)$$

On the other hand, there are non-blades with a zero exterior-square; for example

$$(e_1 + e_1 \wedge e_2) \wedge (e_1 + e_1 \wedge e_2) = 0. \qquad (4.5.12)$$

Theorem 4.5.7 (Non-zero exterior product is equivalent to linear independence for vectors). *Let $A = \{a_1, \dots, a_k\} \subset V$. Then A is linearly independent if and only if*

$$a_1 \wedge \cdots \wedge a_k \neq 0. \qquad (4.5.13)$$

Proof. Assume $a_1 \wedge \cdots \wedge a_k \neq 0$, and consider the equation

$$\sum_{i=1}^{k} \alpha_i a_i = 0, \tag{4.5.14}$$

where $\alpha \in \mathbb{R}^k$. Multiplying this equation by the exterior product with $a_1 \wedge \cdots \wedge a_{j-1}$ from the left, and with $a_{j+1} \wedge \cdots \wedge a_k$ from the right gives, by Theorem 4.5.5,

$$\alpha_j (a_1 \wedge \cdots \wedge a_k) = 0. \tag{4.5.15}$$

Since $a_1 \wedge \cdots \wedge a_k \neq 0$, by the properties of a vector space $\alpha_j = 0$, for all $j \in [1, k]$. Therefore $\{a_1, \dots, a_k\}$ is linearly independent. Assume A is linearly independent. We will prove that $a_1 \wedge \cdots \wedge a_k \neq 0$ by induction. The claim holds trivially for $k = 1$. Assume the claim holds for $k - 1$, where $k > 1$. Since A is independent, by Theorem 2.2.13 $\{a_1, \dots, a_{k-1}\}$ is linearly independent. It follows that $a_1 \wedge \cdots \wedge a_{k-1} \neq 0$. Suppose $a_1 \wedge \cdots \wedge a_k = 0$. TODO. □

Theorem 4.5.8 (Blades can be reshaped). *Let $A = \{a_1, \dots, a_k\} \subset V$ and $B = \{b_1, \dots, b_k\} \subset V$ be linearly independent sets of vectors such that span(A) = span(B). Then there exists a unique $\beta \in \mathbb{R} \setminus \{0\}$, such that*

$$b_1 \wedge \cdots \wedge b_k = \beta (a_1 \wedge \cdots \wedge a_k). \tag{4.5.16}$$

Proof. Since A is a basis for $\operatorname{span}(A)$, and $B \subset \operatorname{span}(A)$, there exists a unique $\alpha_i \in \mathbb{R}^k$ such that

$$b_i = \sum_{j=1}^{k} \alpha_{ij} a_j. \tag{4.5.17}$$

Now

$$\begin{aligned} b_1 \wedge \cdots \wedge b_k &= \left(\sum_{j_1=1}^{k} \alpha_{1j_1} a_{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_k=1}^{k} \alpha_{1j_k} a_{j_k} \right) \\ &= \sum_{j_1=1}^{k} \cdots \sum_{j_k=1}^{k} \alpha_{1j_1} \cdots \alpha_{kj_k} (a_{j_1} \wedge \cdots \wedge a_{j_k}) \\ &= \sum_{J=(j_1,\dots,j_k) \in \sigma(k)} \alpha_{1j_1} \cdots \alpha_{kj_k} (a_{j_1} \wedge \cdots \wedge a_{j_k}). \\ &= \sum_{J=(j_1,\dots,j_k) \in \sigma(k)} (-1)^{\varepsilon(J)} \alpha_{1j_1} \cdots \alpha_{kj_k} (a_1 \wedge \cdots \wedge a_k) \\ &= \det([\alpha_1, \dots, \alpha_k])(a_1 \wedge \cdots \wedge a_k). \end{aligned} \tag{4.5.18}$$

Let $\beta = \det([\alpha_1, \dots, \alpha_k])$. By Theorem 4.5.7, $a_1 \wedge \cdots \wedge a_k \neq 0$, and $b_1 \wedge \cdots \wedge b_k \neq 0$. Therefore $\beta \neq 0$. □

Theorem 4.5.9 (Exterior products of vectors are exactly the blades). *Let $A_k \in \mathcal{Cl}(V)$. Then A_k is a k-blade if and only if there exists $A = \{a_1, \dots, a_k\} \subset V$ such that*

$$A_k = a_1 \wedge \cdots \wedge a_k. \tag{4.5.19}$$

Proof. Assume $A_k = a_1 \cdots a_k$ is a k-blade, where $\{a_1, \ldots, a_k\} \subset V$ is an orthogonal set. If $k = 0$ or $k = 1$, then the claim holds. Assume $k > 1$. By induction on the definition of the wedge product,

$$\begin{aligned} a_1 \cdots a_k &= \langle a_1 \cdots a_k \rangle_k \\ &= a_1 \wedge (a_2 \cdots a_k) \\ &= \cdots \\ &= a_1 \wedge \cdots \wedge a_k. \end{aligned} \tag{4.5.20}$$

Assume $A_k = a_1 \wedge \cdots \wedge a_k$, where $A = \{a_1, \ldots, a_k\} \subset V$. If $A_k = 0$, then A_k is a k-blade by definition. If $A_k \neq 0$, then Theorem 4.5.7 shows that A is linearly independent. By Theorem 4.5.8 we may choose an orthogonal linearly independent set $B = \{b_1, \ldots, b_k\} \subset V$ such that $\operatorname{span}(B) = \operatorname{span}(A)$, and $A_k = b_1 \wedge \cdots \wedge b_k$. Then, by the above, $A_k = b_1 \ldots b_k$. Therefore A_k is a k-blade. $\square$

Theorem 4.5.10 (Exterior product is $\mathbb{N}$-graded-commutative). *If $A_k \in \mathcal{C}l(V)$ is a k-vector, and $B_l \in \mathcal{C}l(V)$ is an l-vector, then $A_k \wedge B_l = (-1)^{kl} B_l \wedge A_k$.*

Proof. By linearity, we only need to prove the result for k-blades. Let $A_k = a_1 \wedge \cdots \wedge a_k$, and $B_k = b_1 \wedge \cdots \wedge b_l$, where $A = \{a_1, \ldots, a_k\} \subset V$, and $B = \{b_1, \ldots, b_l\} \subset V$. Assume $k = 0$, or $l = 0$. Then by Theorem 4.5.4

$$\begin{aligned} A_k \wedge B_l &= A_k B_l \\ &= B_l A_k \\ &= B_l \wedge A_k, \end{aligned} \tag{4.5.21}$$

and the result holds. Assume $k = l = 1$. By Theorem 4.4.3,

$$\begin{aligned} a_1 \wedge b_1 &= \langle a_1 b_1 \rangle_2 \\ &= \widetilde{\widetilde{\langle a_1 b_1 \rangle}}_2 \\ &= (-1)^{\frac{2(2-1)}{2}} \left\langle \widetilde{a_1 b_1} \right\rangle_2 \\ &= -\langle b_1 a_1 \rangle_2 \\ &= -b_1 \wedge a_1, \end{aligned} \tag{4.5.22}$$

and the result holds. Assume $k > 1$, and $l > 0$. Then by associativity and repeated swapping,

$$\begin{aligned} A_k \wedge B_l &= A_{k-1} \wedge a_k \wedge B_l \\ &= (-1)^l A_{k-1} \wedge B_l \wedge a_k \\ &= \cdots \\ &= (-1)^{kl} B_l \wedge A_k. \end{aligned} \tag{4.5.23}$$

$\square$

Theorem 4.5.11 (Exterior product is $\mathbb{Z}_2$-graded-commutative). *Let $A \in \mathcal{C}l(V)$, and $B \in \mathcal{C}l(V)$. Then*

$$
\begin{aligned}
\langle A\rangle_+ \wedge \langle B\rangle_+ &= +\langle B\rangle_+ \wedge \langle A\rangle_+,\\
\langle A\rangle_+ \wedge \langle B\rangle_- &= +\langle B\rangle_- \wedge \langle A\rangle_+,\\
\langle A\rangle_- \wedge \langle B\rangle_+ &= +\langle B\rangle_+ \wedge \langle A\rangle_-,\\
\langle A\rangle_- \wedge \langle B\rangle_- &= -\langle B\rangle_- \wedge \langle A\rangle_-.
\end{aligned}
\tag{4.5.24}
$$

Proof.

$$
\begin{aligned}
\langle A\rangle_+ \wedge \langle B\rangle_+ &= \left(\sum_{k=0}^{\lfloor n/2\rfloor} A_{2k}\right) \wedge \left(\sum_{l=0}^{\lfloor n/2\rfloor} B_{2l}\right)\\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l=0}^{\lfloor n/2\rfloor} A_{2k}\wedge B_{2l}\\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l=0}^{\lfloor n/2\rfloor} (-1)^{(2k)(2l)} B_{2l}\wedge A_{2k}\\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l=0}^{\lfloor n/2\rfloor} B_{2l}\wedge A_{2k}\\
&= \left(\sum_{l=0}^{\lfloor n/2\rfloor} B_{2l}\right) \wedge \left(\sum_{k=0}^{\lfloor n/2\rfloor} A_{2k}\right)\\
&= \langle B\rangle_+ \wedge \langle A\rangle_+,
\end{aligned}
\tag{4.5.25}
$$

where we used Theorem 4.5.10. Similarly for other combinations. □

Theorem 4.5.12 (Exterior product is alternating for odd elements). *Let $A \in \mathcal{C}l(V)$. Then*

$$\langle A\rangle_- \wedge \langle A\rangle_- = 0. \tag{4.5.26}$$

Proof. This is immediate from Theorem 4.5.11. □

Theorem 4.5.13 (Exterior product from geometric product). *Let $a \in V$ and $B \in \mathcal{C}l(V)$. Then*

$$
\begin{aligned}
a \wedge B &= \frac{aB + \widehat{B}a}{2},\\
B \wedge a &= \frac{Ba + a\widehat{B}}{2}.
\end{aligned}
\tag{4.5.27}
$$

Proof. By linearity we only need to prove the result for k-blades. Let $B_k \in \mathcal{C}l(V)$ be a

k-blade. By Theorem 4.4.12,

$$\begin{aligned}
2(a \wedge B_k) &= 2\langle aB_k\rangle_{k+1} \\
&= \left\langle \left(aB_k - \widehat{B_k}a\right) + \left(aB_k + \widehat{B_k a}\right)\right\rangle_{k+1} \\
&= \left\langle aB_k - \widehat{B_k}a\right\rangle_{k+1} + \left\langle aB_k + \widehat{B_k a}\right\rangle_{k+1} \\
&= \left\langle aB_k + \widehat{B_k}a\right\rangle_{k+1} \\
&= aB_k + \widehat{B_k}a.
\end{aligned} \tag{4.5.28}$$

By Theorem 4.5.10,

$$\begin{aligned}
2(B_k \wedge a) &= 2(-1)^k(a \wedge B_k) \\
&= (-1)^k(aB_k + \widehat{B_k}a) \\
&= B_k a + a\widehat{B_k}.
\end{aligned} \tag{4.5.29}$$

□

Theorem 4.5.14 (Grade selection commutes with vector-preserving homomorphisms). *Let $f : G(V) \to G(W)$ be an algebra homomorphism (algebra anti-homomorphism). Then*

$$f(\langle A\rangle_k) = \langle f(A)\rangle_k \tag{4.5.30}$$

if and only if $f(V) \subset W$.

Proof. We will prove the result assuming f is an algebra homomorphism; the proof for the algebra anti-homomorphism is almost identical. Assume $f(V) \subset W$. By linearity we only need prove the result for l-blades. Let $A_l = a_1 \wedge \cdots \wedge a_l \in \mathcal{C}l(V)$ be an l-blade, where $\{a_1, \dots, a_l\} \subset V$. Now

$$\begin{aligned}
f(A_l) &= f(a_1 \wedge \cdots \wedge a_l) \\
&= f(a_1) \wedge \cdots \wedge f(a_l).
\end{aligned} \tag{4.5.31}$$

Then $f(A_l)$ is also an l-blade, since each $f(a_i) \in W$. Assume $l \neq k$. Then $f(\langle A_l\rangle_k) = 0 = \langle f(A_l)\rangle_k$, and the result holds. Assume $l = k$. Then

$$\begin{aligned}
f(\langle A_k\rangle_k) &= f(A_k) \\
&= \langle f(A_k)\rangle_k,
\end{aligned} \tag{4.5.32}$$

and the result holds. Assume $f(\langle A\rangle_k) = \langle f(A)\rangle_k$ holds. Then in particular

$$f(a) = \langle f(a)\rangle_1, \tag{4.5.33}$$

for all $a \in V$. Therefore $f(V) \subset W$. □

Theorem 4.5.15 (Vector-preserving homomorphisms are outermorphisms). *Let $f : \mathcal{C}l(V) \to \mathcal{C}l(W)$ be an algebra homomorphism (anti-homomorphism). If $f(V) \subset W$, then f is an outermorphism (anti-outermorphism).*

Proof. We will prove the result assuming f is an algebra homomorphism; the proof for the algebra anti-homomorphism is almost identical. By linearity we only need prove the result for k-vectors. Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-vector, and an l-vector, respectively. Then by Theorem 4.4.3

$$\begin{aligned} f(A_k \wedge B_l) &= f(\langle A_k B_l \rangle_{k+l}) \\ &= \langle f(A_k B_l) \rangle_{k+l} \\ &= \langle f(A_k) f(B_l) \rangle_{k+l} \\ &= f(A_k) \wedge f(B_l). \end{aligned} \tag{4.5.34}$$

Therefore f is an outermorphism. □

Example 4.5.16 (Outermorphism which is not an algebra homomorphism). There are outermorphisms which are not algebra homomorphisms $\mathcal{C}l(V) \to \mathcal{C}l(W)$. For example, if $f : G(V) \to G(W)$ is an outermorphism such that $f_{|V}$ is not orthogonal, then by Theorem 4.2.4 f is not an algebra homomorphism $\mathcal{C}l(V) \to \mathcal{C}l(W)$.

Theorem 4.5.17 (Non-zero exterior product implies linear independence for non-scalar blades). *Let $\{A_1, \dots, A_k\} \subset \mathcal{C}l(V) \setminus \mathbb{R}$ be a set of non-scalar blades, such that $A_1 \wedge \cdots \wedge A_k \neq 0$. Then $\{A_1, \dots, A_k\}$ is linearly independent.*

Proof. Assume $A_1 \wedge \cdots \wedge A_k \neq 0$, and consider the equation

$$\sum_{i=1}^{k} \alpha_i A_i = 0, \tag{4.5.35}$$

where $\alpha \in \mathbb{R}^k$. Multiplying this equation by the exterior product with $A_1 \wedge \cdots \wedge A_{j-1}$ from the left, and with $A_{j+1} \wedge \cdots \wedge A_k$ from the right gives

$$\alpha_j (A_1 \wedge \cdots \wedge A_k) = 0. \tag{4.5.36}$$

Since $A_1 \wedge \cdots \wedge A_k \neq 0$, by the properties of a vector space $\alpha_j = 0$, for all $j \in [1, k]$. Therefore $\{A_1, \dots, A_k\}$ is linearly independent. □

Example 4.5.18. The converse of Theorem 4.5.7 does not hold in full generality. For example, while $e_1 \wedge e_2$ and $e_2 \wedge e_3$ are linearly independent in $\mathcal{C}l(V)_3$, $(e_1 \wedge e_2) \wedge (e_2 \wedge e_3) = 0$. However, Theorem 4.5.7 shows that the converse holds for vectors.

Remark 4.5.19. The exterior algebra $G(V)$ is algebra-isomorphic to $Cl(V, 0)$, where 0 is the zero function $V^2 \to \mathbb{R}$.

4.6 Blades

Theorem 4.6.1 (Some k-vectors are always blades). *Let V be an n-dimensional vector space. Then 0-vectors, 1-vectors, $(n-1)$-vectors, and n-vectors in $G(V)$ are blades.*

Proof. For 0-vectors and 1-vectors, the result is by definition. For n-vectors the result is obvious, because n-vectors have dimension 1, and so are spanned by a single n-blade.

Let $A = \{a_1, \dots, a_n\} \subset V$ be a basis of V. Then each $(n-1)$-vector $A_{n-1} \in \mathcal{C}l(V)$ is of the form

$$A_{n-1} = \sum_{i=1}^{n} \alpha_i (a_1 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_n), \tag{4.6.1}$$

where $\alpha \in \mathbb{R}^n$, and the check-mark denotes a missing factor. When $n = 1$, the $(n-1)$-vectors are scalars, and so blades. Assume $n \geq 2$. Suppose for a moment that $\alpha_2 \cdots \alpha_{n-1} \neq 0$. We claim that A_{n-1} can be rewritten as

$$A_{n-1} = \frac{1}{\alpha_2 \cdots \alpha_{n-1}} \bigwedge_{i=1}^{n-1} (\alpha_{i+1} a_i + \alpha_i a_{i+1}), \tag{4.6.2}$$

which directly shows that every $(n-1)$-vector is an $(n-1)$-blade. When $n = 2$, the rewriting gives

$$A_{n-1} = \alpha_2 a_1 + \alpha_1 a_2. \tag{4.6.3}$$

Thus the result holds. Assume the result holds for $(n-1)$, where $n > 2$. First notice that

$$\begin{aligned} \alpha_n (a_1 \wedge \cdots \wedge a_{n-1}) &= \left[\frac{1}{\alpha_2 \cdots \alpha_{n-1}} \bigwedge_{i=1}^{n-2} \alpha_i a_{i+1} \right] \wedge (\alpha_n a_{n-1}) \\ &= \left[\frac{1}{\alpha_2 \cdots \alpha_{n-1}} \bigwedge_{i=1}^{n-2} (\alpha_{i+1} a_i + \alpha_i a_{i+1}) \right] \wedge (\alpha_n a_{n-1}). \end{aligned} \tag{4.6.4}$$

Then by induction

$$\begin{aligned} A_{n-1} &= \sum_{i=1}^{n} \alpha_i (a_1 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_n) \\ &= \left[\sum_{i=1}^{n-1} \alpha_i (a_1 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_{n-1}) \right] \wedge a_n + \alpha_n (a_1 \wedge \cdots \wedge a_{n-1}) \\ &= \left[\frac{1}{\alpha_2 \cdots \alpha_{n-2}} \bigwedge_{i=1}^{n-2} (\alpha_{i+1} a_i + \alpha_i a_{i+1}) \right] \wedge a_n + \left[\frac{1}{\alpha_2 \cdots \alpha_{n-1}} \bigwedge_{i=1}^{n-2} (\alpha_{i+1} a_i + \alpha_i a_{i+1}) \right] \wedge (\alpha_n a_{n-1}) \\ &= \left[\frac{1}{\alpha_2 \cdots \alpha_{n-1}} \bigwedge_{i=1}^{n-2} (\alpha_{i+1} a_i + \alpha_i a_{i+1}) \right] \wedge (\alpha_n a_{n-1} + \alpha_{n-1} a_n) \\ &= \frac{1}{\alpha_2 \cdots \alpha_{n-1}} \bigwedge_{i=1}^{n-1} (\alpha_{i+1} a_i + \alpha_i a_{i+1}). \end{aligned} \tag{4.6.5}$$

Suppose now that $\alpha_i = 0$. Since all the remaining basis $(n-1)$-blades contain the factor a_i, it can be factored out by the exterior product; similarly for all a_i for which $\alpha_i = 0$. What remains is a simplified problem of factoring $(n-1-k)$-vectors for a subspace of dimension $(n-k)$, where none of the factors are zero. Since by induction we know how to do this, every $(n-1)$-vector is an $(n-1)$-blade. □

Remark 4.6.2 (Non-blade k-vectors exists if and only if $\dim(V) > 3$). By Theorem 4.6.1, if $\dim(V) \leq 3$, then all k-vectors of $G(V)$ are k-blades. On the other hand, if $\dim(V) > 3$, then $G(V)$ contains k-vectors which are not k-blades; take for example $e_1 \wedge e_2 + e_3 \wedge e_4$, where $\{e_1, \dots, e_4\} \subset V$ is linearly independent.

4.7 Scalar product

The **scalar product** is a function $* : \mathcal{C}l(V)^2 \to \mathbb{R}$ defined by

$$A * B = \left\langle A\widetilde{B} \right\rangle_0. \tag{4.7.1}$$

Theorem 4.7.1 (Scalar product is bilinear). *The scalar product is bilinear.*

Proof. The geometric product is bilinear, and the grade-selection operator is linear. □

Theorem 4.7.2 (Scalar product is symmetric). *The scalar product is symmetric.*

Proof. By linearity, we only need to prove the result for k-vectors. Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-vector, and an l-vector, respectively. By Theorem 4.4.3,

$$\begin{aligned}
A_k * B_l &= \left\langle A_k \widetilde{B_l} \right\rangle_0 \\
&= \left\langle \widetilde{A_k \widetilde{B_l}} \right\rangle_0 \\
&= \left\langle \widetilde{A_k \widetilde{\widetilde{B_l}}} \right\rangle_0 \\
&= \left\langle B_l \widetilde{A_k} \right\rangle_0 \\
&= B_l * A_k.
\end{aligned} \tag{4.7.2}$$

□

Theorem 4.7.3 (Scalar product on vectors). *Let $a, b \in V$. Then*

$$a * b = a \cdot b. \tag{4.7.3}$$

Proof. By Theorem 4.7.2,

$$\begin{aligned}
2(a * b) &= a * b + b * a && (4.7.4) \\
&= \langle ab \rangle_0 + \langle ba \rangle_0 && (4.7.5) \\
&= \langle ab + ba \rangle_0 && (4.7.6) \\
&= 2(a \cdot b). && (4.7.7)
\end{aligned}$$

□

Remark 4.7.4. Theorems 4.7.1, 4.7.2, and 4.7.3 show that the scalar product extends the dot product $\cdot$ in V to a symmetric bilinear form in $\mathcal{C}l(V)$. This makes the theory of bilinear spaces, particular that of finite-dimensional ones, applicable in $\mathcal{C}l(V)$.

Theorem 4.7.5 (Scalar product is zero for blades of different grade). *Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-vector, and an l-vector, respectively. If $k \neq l$, then*

$$A_k * B_l = 0. \tag{4.7.8}$$

Proof. By Theorem 4.4.11, $\left\langle A_k \widetilde{B_l} \right\rangle_0 = 0$. □

Theorem 4.7.6 (Determinant formula for the scalar product of blades). *Let $A_k, B_k \in \mathcal{C}l(V)$ be k-blades such that $A_k = a_1 \cdots a_k$ and $B_k = b_1 \cdots b_k$, where $\{a_1, \dots, a_k\} \subset V$ and $\{b_1, \dots, b_k\} \subset V$ are orthogonal sets of vectors. Then*

$$A_k * B_k = \begin{vmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_k \\ \vdots & \ddots & \vdots \\ a_k \cdot b_1 & \cdots & a_k \cdot b_k \end{vmatrix} \tag{4.7.9}$$

Theorem 4.7.7 (Scalar product for k-versors). *Let $A = a_1 \cdots a_k \in \mathcal{C}l(V)$ be a k-versor, where $\{a_1, \dots, a_k\} \subset V$ is a set of vectors. Then*

$$A * A = \prod_{i=1}^{k} a_i \cdot a_i. \tag{4.7.10}$$

Proof.

$$\begin{aligned} A * A &= \left\langle A \widetilde{A} \right\rangle_0 \\ &= \langle a_1 \cdots a_k a_k \cdots a_1 \rangle_0 \\ &= \prod_{i=1}^{k} a_i^2 \\ &= \prod_{i=1}^{k} a_i \cdot a_i. \end{aligned} \tag{4.7.11}$$

□

Theorem 4.7.8 (Scalar product is positive-definite exactly when the dot product is). *The scalar product is positive-definite (positive semi-definite) if and only if $\cdot$ is positive-definite (positive semi-definite).*

Proof. By linearity, and Theorem 4.7.5, we only need to prove the result for k-blades. Assume $\cdot$ is positive-definite. Let $A_k = a_1 \cdots a_k \in \mathcal{C}l(V)$ be a k-blade, where $\{a_1, \dots, a_k\} \subset V$ is an orthogonal set of vectors. Then $A_k \neq 0$ implies $a_1, \dots, a_k \neq 0$. By Theorem 4.7.7 and positive-definiteness of $\cdot$,

$$A_k * A_k = \prod_{i=1}^{k} a_i \cdot a_i > 0. \tag{4.7.12}$$

Therefore the scalar product is positive-definite. Assume $\cdot$ is not positive-definite. Then there exists an $a \in V \setminus \{0\}$ such that $a \cdot a \leq 0$. By Theorem 4.7.3, $a \cdot a = a * a$. Therefore the scalar product is not positive-definite. The proof for the positive semi-definite case is similar. □

Theorem 4.7.9 (Vector-preserving homomorphisms are orthogonal). *Let $f : \mathcal{C}l(V) \to \mathcal{C}l(W)$ be an algebra homomorphism such that $f(V) \subset W$. Then f is orthogonal.*

Proof. By Theorem 4.4.3,

$$\begin{aligned} f(A) * f(B) &= \left\langle f(A)\widetilde{f(B)} \right\rangle_0 \\ &= \left\langle f(A) f\left(\widetilde{B}\right) \right\rangle_0 \\ &= \left\langle f(A\widetilde{B}) \right\rangle_0 \\ &= f\left(\left\langle A\widetilde{B} \right\rangle_0\right) \\ &= f(A * B) \\ &= A * B. \end{aligned} \tag{4.7.13}$$

Therefore f is orthogonal. □

Remark 4.7.10. Theorem 4.7.9 generalizes Theorem 4.2.4 for the dot product to the scalar product.

Theorem 4.7.11 (Versor inverse is also an inverse of the scalar product). *Let $B \in \mathcal{C}l(V)$ be a k-versor such that $B * B \neq 0$. Then $\frac{B}{B*B}$ is an inverse of B with respect to $*$.*

Proof.

$$B * \frac{B}{B * B} = \frac{B * B}{B * B} = 1 = \frac{B * B}{B * B} = \frac{B}{B * B} * B. \tag{4.7.14}$$

□

Remark 4.7.12. The inverse of an invertible k-versor $B \in \mathcal{C}l(V)$ with respect to the scalar product, given in Theorem 4.7.11, equals the inverse of B with respect to the geometric product, given in Theorem 4.3.2, since $B * B = \left\langle B\widetilde{B} \right\rangle_0 = B\widetilde{B}$. However, the inverse for the scalar product is not unique (scalar product is not associative). For example, if $a, b \in \mathcal{C}l(\mathbb{R}^{2,0})$, with $a = e_1$, $b = e_1 + \alpha e_2$, and $\alpha \in \mathbb{R}$, then $a \cdot b = 1 = b \cdot a$, no matter what α is.

4.8 Contraction

The **left contraction** is a function $\rfloor : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A \rfloor B = \sum_{k=0}^{n} \sum_{l=0}^{n} \langle A_k B_l \rangle_{l-k}. \tag{4.8.1}$$

The **right contraction** is a function $\lfloor : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A \lfloor B = \sum_{k=0}^{n} \sum_{l=0}^{n} \langle A_k B_l \rangle_{k-l}. \tag{4.8.2}$$

The **inner product** is a function $\cdot : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A \cdot B = \sum_{k=0}^{n} \sum_{l=0}^{n} \langle A_k B_l \rangle_{|k-l|}. \tag{4.8.3}$$

Remark 4.8.1 (Using both contractions is unnecessary, but provides symmetry). It is enough to use either the left or the right contraction in derivations. For this reason, it is often just the left contraction which is defined, and then it is called the contraction. However, using both reveals symmetry in equations and is sometimes more convenient in derivations; this is why we define them both. This is similar to the use of both less-than and greater-than.

Theorem 4.8.2 (Contraction is bilinear). *Contraction is bilinear.*

Proof. Geometric product is bilinear, and the grade selection operator is linear. □

Remark 4.8.3. Contraction is not associative or commutative.

Theorem 4.8.4 (Right contraction to left contraction). *Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-vector, and an l-vector, respectively. Then*

$$A_k \lfloor B_l = (-1)^{l(k+1)} B_l \rfloor A_k. \tag{4.8.4}$$

Proof. By Theorem 4.4.3,

$$\begin{aligned}
A_k \lfloor B_l &= \langle A_k B_l \rangle_{k-l} \\
&= \langle \widetilde{\widetilde{A_k B_l}} \rangle_{k-l} \\
&= (-1)^{\frac{(k-l)(k-l-1)}{2}} \left\langle \widetilde{A_k B_l} \right\rangle_{k-l} \\
&= (-1)^{\frac{(k-l)(k-l-1)}{2}} \left\langle \widetilde{B_l} \widetilde{A_k} \right\rangle_{k-l} \\
&= (-1)^{\frac{(k-l)(k-l-1)}{2}} (-1)^{\frac{l(l-1)}{2}} (-1)^{\frac{k(k-1)}{2}} \langle B_l A_k \rangle_{k-l} \\
&= (-1)^{\frac{(k-l)(k-l-1)}{2}} (-1)^{\frac{l(l-1)}{2}} (-1)^{\frac{k(k-1)}{2}} B_l \rfloor A_k \\
&= (-1)^{k^2+l^2-k-kl} B_l \rfloor A_k \\
&= (-1)^{k+l-k-kl} B_l \rfloor A_k \\
&= (-1)^{l(1-k)} B_l \rfloor A_k \\
&= (-1)^{l(k+1)} B_l \rfloor A_k,
\end{aligned} \tag{4.8.5}$$

where we used the identities $(-1)^{k^2} = (-1)^k = (-1)^{-k} = (-1)^{k+2l}$. □

Theorem 4.8.5 (Contraction of a scalar). *Let $\alpha \in \mathbb{R}$ and $B \in \mathcal{C}l(V)$. Then*

$$\begin{aligned}
\alpha \rfloor B &= \alpha B \\
B \lfloor \alpha &= \alpha B.
\end{aligned} \tag{4.8.6}$$

Proof. By linearity we only need to prove the result for the k-vectors. Let $B_k \in \mathcal{C}l(V)$ be a k-vector. Now

$$\begin{aligned} \alpha \rfloor B_k &= \langle \alpha B_k \rangle_k \\ &= \alpha B_k. \end{aligned} \tag{4.8.7}$$

By Theorem 4.8.4,

$$\begin{aligned} B_k \lfloor \alpha &= (-1)^0 \alpha \rfloor B_k \\ &= \alpha B_k. \end{aligned} \tag{4.8.8}$$

$\square$

Theorem 4.8.6 (Contraction on a scalar). *Let $A \in \mathcal{C}l(V)$, and $\beta \in \mathbb{R}$. Then*

$$\begin{aligned} A \rfloor \beta &= \beta \langle A \rangle_0 \\ \beta \lfloor A &= \beta \langle A \rangle_0. \end{aligned} \tag{4.8.9}$$

Proof. By linearity we only need to prove the result for the k-vectors. Let $A_k \in \mathcal{C}l(V)$ be a k-vector. Now

$$\begin{aligned} A_k \rfloor \beta &= \langle A_k \beta \rangle_{-k} \\ &= \beta \langle A_k \rangle_{-k} \\ &= \beta \langle A_k \rangle_0. \end{aligned} \tag{4.8.10}$$

By Theorem 4.8.4,

$$\begin{aligned} \beta \lfloor A_k &= (-1)^k A_k \rfloor \beta \\ &= (-1)^k \beta \langle A_k \rangle_0 \\ &= \beta \langle A_k \rangle_0, \end{aligned} \tag{4.8.11}$$

since the result is non-zero only if $k = 0$. $\square$

Theorem 4.8.7 (Contraction between k-vectors of the same grade). *Let $A_k, B_k \in \mathcal{C}l(V)$ be k-vectors. Then*

$$\begin{aligned} A_k \rfloor B_k &= A_k * \widetilde{B_k} \\ B_k \lfloor A_k &= A_k * \widetilde{B_k}. \end{aligned} \tag{4.8.12}$$

Proof.

$$\begin{aligned} A_k \rfloor B_k &= \langle A_k B_k \rangle_0 \\ &= A_k * \widetilde{B_k}. \end{aligned} \tag{4.8.13}$$

By Theorem 4.8.4,

$$\begin{aligned} B_k \lfloor A_k &= (-1)^{k(k+1)} A_k \rfloor B_k \\ &= A_k * \widetilde{B_k}. \end{aligned} \tag{4.8.14}$$

$\square$

Theorem 4.8.8 (Contraction from the geometric product). *Let $a \in V$ and $B \in \mathcal{C}l(V)$. Then*

$$\begin{aligned} a \rfloor B &= \frac{aB - \widehat{B}a}{2} \\ B \lfloor a &= \frac{Ba - a\widehat{B}}{2} \end{aligned} \tag{4.8.15}$$

Proof. By linearity we only need to prove the result for k-blades. Let $B_k \in \mathcal{C}l(V)$ be a k-blade. By Theorem 4.4.12,

$$\begin{aligned} 2(a \rfloor B_k) &= 2\langle aB_k \rangle_{k-1} \\ &= \left\langle \left(aB_k - \widehat{B_k}a \right) + \left(aB_k + \widehat{B_k a} \right) \right\rangle_{k-1} \\ &= \left\langle aB_k - \widehat{B_k}a \right\rangle_{k-1} + \left\langle aB_k + \widehat{B_k a} \right\rangle_{k-1} \\ &= \left\langle aB_k - \widehat{B_k}a \right\rangle_{k-1} \\ &= aB_k - \widehat{B_k}a. \end{aligned} \tag{4.8.16}$$

By Theorem 4.8.4,

$$\begin{aligned} 2(B_k \lfloor a) &= (-1)^{k+1} 2(a \rfloor B_k) \\ &= (-1)^{k+1}(aB_k - \widehat{B_k}a) \\ &= B_k a - a\widehat{B_k}. \end{aligned} \tag{4.8.17}$$

□

Theorem 4.8.9 (First duality). *Let $A, B, C \in \mathcal{C}l(V)$. Then*

$$\begin{aligned} (A \wedge B) \rfloor C &= A \rfloor (B \rfloor C) \\ C \lfloor (B \wedge A) &= (C \lfloor B) \lfloor A. \end{aligned} \tag{4.8.18}$$

Proof. By linearity we only need to prove the result for k-blades. Let $A_k = a_1 \wedge \cdots \wedge a_k \in \mathcal{C}l(V)$ be a k-blade, where $\{a_1, \ldots, a_k\} \subset V$, and $C_m \in \mathcal{C}l(V)$ be an m-blade. First note that

$$\left\langle a_k \wedge (\widehat{A_{k-1} C_m}) \right\rangle_{m-k} = 0. \tag{4.8.19}$$

This is trivially true when $k > m$. If $k \leq m$, then the lowest grade is given by Theorem 4.4.11 as $m - k + 2$, which is greater than the selected grade $m - k$. By Theorems 4.5.13

and 4.8.8,

$$
\begin{aligned}
A_k \rfloor C_m &= \langle A_k C_m \rangle_{m-k} \\
&= \langle (A_{k-1} \wedge a_k) C_m \rangle_{m-k} \\
&= \frac{1}{2}\Big\langle (A_{k-1} a_k + a_k \widehat{A_{k-1}}) C_m \Big\rangle_{m-k} \\
&= \frac{1}{2}\Big\langle A_{k-1} a_k C_m - A_{k-1}\widehat{C_m} a_k + A_{k-1}\widehat{C_m} a_k + a_k \widehat{A_{k-1}} C_m \Big\rangle_{m-k} \\
&= \Big\langle A_{k-1}(a_k \rfloor C_m) + (a_k \wedge (\widehat{A_{k-1}} C_m)) \Big\rangle_{m-k} \\
&= \langle A_{k-1}(a_k \rfloor C_m) \rangle_{(m-1)-(k-1)} \\
&= A_{k-1} \rfloor (a_k \rfloor C_m) \\
&= \dots \\
&= a_1 \rfloor (a_2 \rfloor \cdots (a_k \rfloor C_m) \cdots).
\end{aligned}
\tag{4.8.20}
$$

Let $B_l = a_1 \wedge \cdots \wedge b_l \in \mathcal{C}l(V)$ be an l-blade, where $\{b_1, \dots, b_l\} \subset V$. Then

$$
\begin{aligned}
(A_k \wedge B_l) \rfloor C_m &= a_1 \rfloor (\cdots (a_k \rfloor (b_1 \rfloor \cdots (b_l \rfloor C_m) \cdots) \\
&= a_1 \rfloor (\cdots (a_k \rfloor (B_l \rfloor C_m)) \cdots) \\
&= A_k \rfloor (B_l \rfloor C_m).
\end{aligned}
\tag{4.8.21}
$$

By Theorem 4.8.4,

$$
\begin{aligned}
C_m \lfloor (B_l \wedge A_k) &= (-1)^{(l+k)(m+1)} (B_l \wedge A_k) \rfloor C_m \\
&= (-1)^{(l+k)(m+1)} (-1)^{lk} (A_k \wedge B_l) \rfloor C_m \\
&= (-1)^{(l+k)(m+1)} (-1)^{lk} A_k \rfloor (B_l \rfloor C_m) \\
&= (-1)^{(l+k)(m+1)} (-1)^{lk} (-1)^{l(m+1)} A_k \rfloor (C_m \lfloor B_l) \\
&= (-1)^{(l+k)(m+1)} (-1)^{lk} (-1)^{l(m+1)} (-1)^{k(m+l+1)} (C_m \lfloor B_l) \lfloor A_k \\
&= (-1)^{(l+k)(m+1)} (-1)^{l(m+1)} (-1)^{k(m+1)} (C_m \lfloor B_l) \lfloor A_k \\
&= (-1)^{(l+k)(m+1)} (-1)^{(l+k)(m+1)} (C_m \lfloor B_l) \lfloor A_k \\
&= (C_m \lfloor B_l) \lfloor A_k.
\end{aligned}
\tag{4.8.22}
$$

□

Theorem 4.8.10 (Contraction of a vector with a geometric product). *Let $a \in V$, and $B_1, \dots, B_k \in \mathcal{C}l(V)$. Then*

$$
\begin{aligned}
a \rfloor (B_1 \cdots B_k) &= \sum_{i=1}^{k} \widehat{B_1} \cdots \widehat{B_{i-1}} (a \rfloor B_i) B_{i+1} \cdots B_k \\
(B_1 \cdots B_k) \lfloor a &= \sum_{i=1}^{k} B_1 \cdots B_{i-1} (B_i \lfloor a) \widehat{B_{i+1}} \cdots \widehat{B_k}.
\end{aligned}
\tag{4.8.23}
$$

Proof. The result holds trivially for $k = 1$. Assume $k > 1$. By Theorem 4.8.8 and induction,

$$\begin{aligned}
2(a \rfloor (B_1 \cdots B_k)) &= a(B_1 \cdots B_k) - \widehat{B_1 \cdots B_k} a \\
&= (aB_1 \cdots B_{k-1})B_k - (\widehat{B_1 \cdots B_{k-1}} a)B_k + (\widehat{B_1 \cdots B_{k-1}})aB_k - (\widehat{B_1 \cdots B_{k-1}})\widehat{B_k} a \\
&= (aB_1 \cdots B_{k-1} - \widehat{B_1 \cdots B_{k-1}} a)B_k + (\widehat{B_1 \cdots B_{k-1}})(aB_k - \widehat{B_k} a) \\
&= 2(a \rfloor (B_1 \cdots B_{k-1}))B_k + 2(\widehat{B_1 \cdots B_{k-1}})(a \rfloor B_k) \\
&= \dots \\
&= 2 \sum_{i=1}^{k} \widehat{B_1} \cdots \widehat{B_{i-1}} (a \rfloor B_i) B_{i+1} \cdots B_k.
\end{aligned} \tag{4.8.24}$$

Again, by Theorem 4.8.8,

$$\begin{aligned}
(B_1 \cdots B_k) \lfloor a &= \widehat{a} \rfloor (\widehat{B_1 \cdots B_k}) \\
&= \sum_{i=1}^{k} B_1 \cdots B_{i-1} (\widehat{a} \rfloor \widehat{B_i}) \widehat{B_{i+1}} \cdots \widehat{B_k} \\
&= \sum_{i=1}^{k} B_1 \cdots B_{i-1} (B_i \lfloor a) \widehat{B_{i+1}} \cdots \widehat{B_k}.
\end{aligned} \tag{4.8.25}$$

□

Remark 4.8.11 (Varying definitions for the contraction). The literature uses varying definitions in place of the (left) contraction. These definitions agree in common cases, but differ in corner cases. It has been argued in [3] that the definition we give is the most natural one, and also produces more information than the others. We shall now review some of these commonly used definitions. The **fat-dot product** [2] is a function $\bullet_D : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A \bullet_D B = \sum_{k=0}^{n} \sum_{l=0}^{n} \langle A_k B_l \rangle_{|l-k|}. \tag{4.8.26}$$

If $A_k, B_l \in \mathcal{C}l(V)$ are a k-vector, and an l-vector, respectively, then

$$A_k \bullet_D B_l = \begin{cases} A_k \rfloor B_l & \text{if } k \leq l \\ A_k \lfloor B_l & \text{if } k > l. \end{cases} \tag{4.8.27}$$

Since $A_k \rfloor B_l = 0$ for $k > l$, it may at first sound like a good idea to fill in the zeros with something more useful. However, this misses the fact that the 0 result is itself geometrically meaningful; it means that not all of A_k is contained in B_l. Therefore the fat-dot product produces less information than the left or the right contraction. This is especially problematic in a computer implementation of geometric algebra, where the grades of the blades may be impossible to know in advance because of rounding errors.

This results in the need to branch based on the grades of the blades to recover the lost information. The **Hestenes's inner product** [4] is a function $\bullet_H : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A \bullet_H B = \sum_{k=1}^{n} \sum_{l=1}^{n} \langle A_k B_l \rangle_{|l-k|}. \tag{4.8.28}$$

It is otherwise equal to the fat-dot product, but it is zero when either argument is a 0-blade. In addition to missing the geometric significance of the 0-result, the Hestenes inner product misses the geometric signifance of the left contraction with a scalar (a 0-blade). Again that information must be recovered by branching on the grades of the blades. Suppose $k, l > 0$. Then

$$\begin{aligned} A_k \rfloor B_l = A_k \bullet_D B_l = A_k \bullet_H B_l, \text{ if } k \leq l, \text{ and} \\ A_k \lfloor B_l = A_k \bullet_D B_l = A_k \bullet_H B_l, \text{ if } k > l, \end{aligned} \tag{4.8.29}$$

which shows that the definitions agree on most of the cases. Only the left contraction and the right contraction produce the full amount of information.

4.9 Interplay

This sections contains theorems which relate the different products together.

Theorem 4.9.1 (Geometric decomposition). *Let $a \in V$ and $B \in \mathcal{C}l(V)$. Then*

$$\begin{aligned} aB &= a \rfloor B + a \wedge B \\ Ba &= B \lfloor a + B \wedge a. \end{aligned} \tag{4.9.1}$$

Proof. By linearity, we only need to prove the result for k-vectors. Let $B_k \in \mathcal{C}l(V)$ be a k-vector. By Theorem 4.4.10,

$$\begin{aligned} aB_k &= \langle aB_k \rangle_{k-1} + \langle aB_k \rangle_{k+1} \\ &= a \rfloor B_k + a \wedge B_k. \end{aligned} \tag{4.9.2}$$

□

Theorem 4.9.2 (Exterior product of a vector with a geometric product). *Let $a \in V$, and $B_1, \dots, B_k \in \mathcal{C}l(V)$. Then*

$$\begin{aligned} a \wedge (B_1 \cdots B_k) &= \left[\sum_{i=1}^{k-1} \left(\widehat{B_1} \cdots \widehat{B_{i-1}} \right) (a \rfloor B_i)(B_{i+1} \cdots B_k) \right] + \left(\widehat{B_1} \cdots \widehat{B_{k-1}} \right) (a \wedge B_k) \\ (B_1 \cdots B_k) \wedge a &= (B_1 \wedge a) \left(\widehat{B_2} \cdots \widehat{B_k} \right) + \left[\sum_{i=2}^{k} (B_1 \cdots B_{i-1})(B_i \lfloor a) \left(\widehat{B_{i+1}} \cdots \widehat{B_k} \right) \right]. \end{aligned} \tag{4.9.3}$$

Proof. The result holds trivially for $k = 1$. Assume $k > 1$. □

Theorem 4.9.3 (Contraction of a vector with an exterior product). *Let $a \in V$, and $B_1, \dots, B_k \in \mathcal{C}l(V)$. Then*

$$\begin{aligned} a \rfloor (B_1 \wedge \cdots \wedge B_k) &= \sum_{i=1}^{k} \left(\widehat{B_1} \wedge \cdots \wedge \widehat{B_{i-1}} \right) \wedge (a \rfloor B_i) \wedge (B_{i+1} \wedge \cdots \wedge B_k) \\ (B_1 \wedge \cdots \wedge B_k) \lfloor a &= \sum_{i=1}^{k} (B_1 \wedge \cdots \wedge B_{i-1}) \wedge (B_i \lfloor a) \wedge \left(\widehat{B_{i+1}} \wedge \cdots \wedge \widehat{B_k} \right). \end{aligned} \tag{4.9.4}$$

Proof. Theorem 4.8.10 works in particular for k-blades. □

Theorem 4.9.4 (Geometric inverse is an inverse of contraction). *Let $A_k \in \mathcal{C}l(V)$ be an invertible k-blade. Then*

$$\begin{aligned} A_k \rfloor A_k^{-1} = 1 = A_k^{-1} \rfloor A_k \\ A_k^{-1} \lfloor A_k = 1 = A_k \lfloor A_k^{-1}. \end{aligned} \tag{4.9.5}$$

Proof. By the definition of the left contraction

$$\begin{aligned} A_k \rfloor A_k^{-1} &= \left\langle A_k A_k^{-1} \right\rangle_0 \\ &= 1 \\ &= \left\langle A_k^{-1} A_k \right\rangle_0 \\ &= A_k^{-1} \rfloor A_k. \end{aligned} \tag{4.9.6}$$

Similarly for the right contraction. □

Remark 4.9.5. As with the scalar product, the contraction inverse given in Theorem 4.9.4 is not unique (contraction is not associative). The same example applies as with the scalar product.

4.10 Span

The **span** is a function $\mathcal{C}l(V) \to P(V)$ defined by

$$\operatorname{span}(A) = \{x \in V : x \wedge A = 0\}. \tag{4.10.1}$$

A multi-vector $A \in \mathcal{C}l(V)$ is called **reducible**, if there exists $a \in V$, and $B \in \mathcal{C}l(V)$, such that

$$A = a \wedge B. \tag{4.10.2}$$

A multi-vector is called **irreducible**, if it is not reducible.

Theorem 4.10.1 (Span is a subspace). *Let $A \in \mathcal{C}l(V)$. Then span(A) is a subspace of V.*

Proof. Let $a, b \in \operatorname{span}(A)$, and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} (\alpha a + \beta b) \wedge A &= \alpha(a \wedge A) + \beta(b \wedge A) \\ &= 0. \end{aligned} \tag{4.10.3}$$

Thus $\alpha a + \beta b \in \operatorname{span}(A)$. Therefore span$(A)$ is a subspace of V. □

Theorem 4.10.2 (Span is intersection of grade spans). *Let $A \in \mathcal{C}l(V)$. Then*

$$\text{span}(A) = \bigcap_{k=0}^{n-1} \text{span}(A_k). \tag{4.10.4}$$

Proof. Assume $a \in \text{span}(A)$. Then

$$\begin{aligned} a \wedge A &= a \wedge \left(\sum_{k=0}^{n} A_k \right) \\ &= \sum_{k=0}^{n} a \wedge A_k. \end{aligned} \tag{4.10.5}$$

Suppose this sum has non-zero terms. These terms are linearly independent, since they are k-vectors of different grade. By definition they can not then sum to zero; a contradiction. Therefore, if $a \wedge A = 0$, then

$$a \wedge A_k = 0, \tag{4.10.6}$$

for all $k \in [0, n]$. Since $a \wedge A_n = 0$ always holds, we may sharpen this to $k \in [0, n-1]$. Therefore $a \in \bigcap_{k=0}^{n-1} \text{span}(A_k)$. Assume $a \in \bigcap_{k=0}^{n-1} \text{span}(A_k)$. Then $a \wedge A_k = 0$, for all $k \in [0, n-1]$, and it follows that $a \wedge A = 0$. Therefore $a \in \text{span}(A)$. □

Theorem 4.10.3 (Invertible outermorphisms preserve span). *Let $A \in \mathcal{C}l(V)$, and $f : \mathcal{C}l(V) \to \mathcal{C}l(V)$ be an invertible outermorphism. Then*

$$\text{span}(f(A)) = f_{|V}(\text{span}(A)). \tag{4.10.7}$$

Proof.

$$\begin{aligned} \text{span}(f(A)) &= \{x \in V : x \wedge f(A) = 0\} \\ &= \{x \in V : f(f^{-1}(x)) \wedge f(A) = 0\} \\ &= \{x \in V : f(f^{-1}(x) \wedge A) = 0\} \\ &= \{x \in V : f^{-1}(x) \wedge A = 0\} \\ &= f_{|V}(\text{span}(A)). \end{aligned} \tag{4.10.8}$$

□

Theorem 4.10.4 (The common involutions preserve span). *Let $A \in \mathcal{C}l(V)$. Then*

$$\begin{aligned} \text{span}\left(\widehat{A}\right) &= \text{span}(A), \\ \text{span}\left(\widetilde{A}\right) &= \text{span}(A), \\ \text{span}\left(\overline{A}\right) &= \text{span}(A). \end{aligned} \tag{4.10.9}$$

Proof. Since grade involution, reversion, and conjugation are all invertible outermorphisms, by Theorem 4.10.3

$$\begin{aligned} \text{span}\left(\widehat{A}\right) &= -\text{span}(A) = \text{span}(A), \\ \text{span}\left(\widetilde{A}\right) &= \text{span}(A), \\ \text{span}\left(\overline{A}\right) &= \text{span}\left(\widetilde{\widehat{A}}\right) = \text{span}(A). \end{aligned} \tag{4.10.10}$$

□

Theorem 4.10.5 (Span by vectors). *Let $A_k = a_1 \wedge \cdots \wedge a_k \in \mathcal{C}l(V)$ be a k-blade, where $\{a_1, \ldots, a_k\} \subset V$, and $A_k \neq 0$. Then*

$$\mathit{span}(A_k) = \mathit{span}(\{a_1, \ldots, a_k\}). \tag{4.10.11}$$

Proof. Since $A_k \neq 0$, $\{a_1, \ldots, a_k\}$ is linearly independent by Theorem 4.5.7. Let $x \in V$. Then

$$\begin{aligned} & x \in \operatorname{span}(A_k) \\ \Leftrightarrow & x \wedge A_k = 0 \\ \Leftrightarrow & x \wedge a_1 \wedge \cdots \wedge a_k = 0. \end{aligned} \tag{4.10.12}$$

This is equivalent to $\{x\} \cup \{a_1, \ldots, a_k\}$ being linearly dependent by Theorem 4.5.7, which is equivalent to $x \in \operatorname{span}(\{a_1, \ldots, a_k\})$ by Theorem 2.2.23 . □

Example 4.10.6. Theorem 4.10.5 does not hold when $A_k = 0$. For example, consider $B = \{e_1, 2e_1, \ldots, ne_1\} \subset V \subset \mathcal{C}l(V)_n$, for $n > 1$. Then $\operatorname{span}(B) = \operatorname{span}(e_1)$, but $\operatorname{span}(A_k) = \operatorname{span}(0) = V$.

Example 4.10.7. For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\operatorname{span}(\alpha) = \{0\}$. However, $\operatorname{span}(0) = V$.

Theorem 4.10.8 (Blade factorizability is equivalent to non-trivial span). *Let $A_k \in \mathcal{C}l(V) \setminus \{0\}$ be a k-blade, and $a \in V \setminus \{0\}$. Then $a \in \mathit{span}(A_k)$ if and only if there exists a $(k-1)$-blade $A_{k-1} \in \mathcal{C}l(V)$ such that*

$$A_k = a \wedge A_{k-1}. \tag{4.10.13}$$

Proof. Assume $a \in \operatorname{span}(A_k)$. By Theorem 4.5.9, there exists a set $A = \{a_1, \ldots, a_k\} \subset V$ such that

$$A_k = a_1 \wedge \cdots \wedge a_k. \tag{4.10.14}$$

Since $A_k \neq 0$, by Theorem 4.5.7 A is linearly independent. Since $a \in \operatorname{span}(A_k)$, by Theorem 4.10.5 $a \in \operatorname{span}(A)$. Let $B = \{a, b_2, \ldots, b_k\} \subset V$ be linearly independent such that $\operatorname{span}(B) = \operatorname{span}(A)$. Then by Theorem 4.5.8 there exists a unique $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned} A_k &= \beta(a \wedge b_2 \wedge \cdots \wedge b_k) \\ &= a \wedge ((\beta b_2) \wedge \cdots \wedge b_k). \end{aligned} \tag{4.10.15}$$

We may then choose $A_{k-1} = (\beta b_2) \wedge \cdots \wedge b_k$. Assume there exists A_{k-1} such that $A_k = a \wedge A_{k-1}$. Then

$$\begin{aligned} a \wedge A_k &= (a \wedge a) \wedge A_{k-1} \\ &= 0. \end{aligned} \tag{4.10.16}$$

Therefore $a \in \operatorname{span}(A_k)$. □

Theorem 4.10.9 (Irreducible multi-vectors have trivial span). *Let $A \in \mathcal{C}l(V)$ be irreducible. Then*

$$\mathit{span}(A) = \{0\}. \tag{4.10.17}$$

Proof. □

Theorem 4.10.10 (Span of k-vectors). *Let $A_k \in \mathcal{C}l(V) \setminus \{0\}$ be a k-vector, $B_l \in \mathcal{C}l(V)$ be an l-blade, and $C_m \in \mathcal{C}l(V)$ be an irreducible m-vector, such that*

$$A_k = B_l \wedge C_m. \tag{4.10.18}$$

Then span(A_k) = span(B_l).

Proof. Let $n = \dim(V)$. The claim is trivial for $m = 0$. Assume $m \geq 1$. By Theorem 4.6.1 $n \geq 4$, and $k \in [2, n-2]$. Assume $x \in \text{span}(B_l)$. Then $x \wedge B_l = 0$, and

$$\begin{aligned} x \wedge (B_l \wedge C_m) &= (x \wedge B_l) \wedge C_m \\ &= 0. \end{aligned} \tag{4.10.19}$$

Therefore $x \in \text{span}(A_k)$. Assume $x \notin \text{span}(B_l)$. Then $x \wedge B_l$ is a non-zero $(l+1)$-blade. Since C_m is irreducible, it can not have □

Theorem 4.10.11. *Let $A \in \mathcal{C}l(V) \setminus \{0\}$. Then $a \in$ span$(A) \setminus \{0\}$ if and only if there exists $B \in \mathcal{C}l(V)$ such that*

$$A = a \wedge B. \tag{4.10.20}$$

Proof. Let $A = C_k \wedge D$, where $C_k \in \mathcal{C}l(V)$ is a k-blade of maximal grade, and $D \in \mathcal{C}l(V)$. Then D must contain at least one grade D_l which can not factor out a non-zero vector. By Theorem 4.10.8 span$(D_l) = \{0\}$. By Theorem 4.10.2, span$(D) = \{0\}$. It follows that span(A) = span(C_k). The result follows from Theorem 4.10.8. □

Theorem 4.10.12 (Span of non-zero exterior product is direct sum of factor spans). *Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-blade, and an l-blade, respectively, such that $A_k \wedge B_l \neq 0$. Then*

$$\text{span}(A_k \wedge B_l) = \text{span}(A_k) \dotplus \text{span}(B_l). \tag{4.10.21}$$

Proof. Decompose $A_k = a_1 \wedge \cdots \wedge a_k$, and $B_l = b_1 \wedge \cdots \wedge b_l$, where $A = \{a_1, \dots, a_k\} \subset V$, and $B = \{b_1, \dots, b_l\} \subset V$. Since $A_k \wedge B_l \neq 0$, by Theorem 4.5.7 $A \cup B$ is linearly independent. By Theorem 4.10.5,

$$\begin{aligned} \text{span}(A_k \wedge B_l) &= \text{span}(\{a_1, \dots, a_k, b_1, \dots, b_l\}) \\ &= \text{span}(\{a_1, \dots, a_k\}) \dotplus \text{span}(\{b_1, \dots, b_l\}) \\ &= \text{span}(A_k) \dotplus \text{span}(B_l). \end{aligned} \tag{4.10.22}$$

□

Theorem 4.10.13 (Second duality). *Let $A_k, C_m \in \mathcal{C}l(V)$ be a k-blade, and an m-blade, respectively, such that span$(A_k) \subset$ span(C_m), and $B \in \mathcal{C}l(V)$. Then*

$$A_k \wedge (B \rfloor C_m) = (A_k \rfloor B) \rfloor C_m. \tag{4.10.23}$$

Proof. By linearity we only need to prove the result for blades. Let $A_k = a_1 \wedge \cdots \wedge a_k \in \mathcal{C}l(V)$ be a k-blade, where $\{a_1, \ldots, a_k\} \subset V$, $B_l = b_1 \wedge \cdots \wedge b_l$ be an l-blade, where $\{b_1, \ldots, b_l\} \subset V$, and $C_m = c_1 \wedge \cdots \wedge c_m$ be an m-blade, where $\{c_1, \ldots, c_m\} \subset V$. Assume $A_k \in \mathbb{R}$. Then the result holds trivially. Assume $C_m \in \mathbb{R}$, $C_m \neq 0$. Then, because $\operatorname{span}(A_k) \subset \operatorname{span}(C_m)$, also $A_k \in \mathbb{R}$, and thus the result holds. On the other hand, if $C_m = 0$, then the result holds trivially. Assume $B_l \in \mathbb{R}$, $B_l \neq 0$. Since we may now assume $A_k \notin \mathbb{R}$, $A_k \rfloor B_l = 0$. On the other hand, since $\operatorname{span}(A_k) \subset \operatorname{span}(C_m)$, $A_k \wedge C_m = 0$, and thus the result holds. The result holds trivially for $B_l = 0$. Assume from now on that none of A_k, B_l, or C_m is a scalar. We will prove the result by repeated induction. Assume $k = l = m = 1$, and $\operatorname{span}(a_k) \subset \operatorname{span}(c_m)$. Then $a_k = \alpha c_m$. Now

$$\begin{aligned} a_k \wedge (b_l \rfloor c_m) &= c_m \wedge (b_l \rfloor a_k) \\ &= c_m (b_l \rfloor a_k) \\ &= c_m (a_k \rfloor b_l) \\ &= (a_k \rfloor b_l) c_m \\ &= (a_k \rfloor b_l) \rfloor c_m. \end{aligned} \tag{4.10.24}$$

Assume the result holds for $k = 1$, $l = 1$, and $m - 1$, with $m > 1$, and that $\operatorname{span}(a_k) \subset \operatorname{span}(C_m)$. Then by Theorem 4.10.12 either $\operatorname{span}(a_k) \subset \operatorname{span}(C_{m-1})$, or $\operatorname{span}(a_k) \subset \operatorname{span}(c_m)$. Assume the former; then $a_k \wedge C_{m-1} = 0$, which by Theorem 4.10.4 is equivalent to $a_k \wedge \widehat{C_{m-1}} = 0$. Now

$$\begin{aligned} a_k \wedge (b_l \rfloor C_m) &= a_k \wedge (b_l \rfloor (C_{m-1} \wedge c_m)) \\ &= a_k \wedge \Big((b_l \rfloor C_{m-1}) \wedge c_m + \Big(\widehat{C_{m-1}} \wedge (b_l \rfloor c_m) \Big) \Big) \\ &= a_k \wedge (b_l \rfloor C_{m-1}) \wedge c_m \\ &= ((a_k \rfloor b_l) \rfloor C_{m-1}) \wedge c_m \\ &= (a_k \rfloor b_l)(C_{m-1} \wedge c_m) \\ &= (a_k \rfloor b_l)(C_m). \end{aligned} \tag{4.10.25}$$

Assume instead that $\operatorname{span}(a_k) \subset \operatorname{span}(c_m)$; then $a_k \wedge c_m = 0$, and in particular $a_k = \alpha c_m$. Now

$$\begin{aligned} a_k \wedge (b_l \rfloor C_m) &= a_k \wedge \Big((b_l \rfloor C_{m-1}) \wedge c_m + \Big(\widehat{C_{m-1}} \wedge (b_l \rfloor c_m) \Big) \Big) \\ &= a_k \wedge \widehat{C_{m-1}} \wedge (b_l \rfloor c_m) \\ &= c_m \wedge \widehat{C_{m-1}} \wedge (b_l \rfloor a_k) \\ &= C_{m-1} \wedge c_m \wedge (a_k \rfloor b_l) \\ &= C_m \wedge (a_k \rfloor b_l) \\ &= (a_k \rfloor b_l) \wedge C_m \\ &= (a_k \rfloor b_l) \rfloor C_m. \end{aligned} \tag{4.10.26}$$

Assume the result holds for $k = 1$, and $l - 1$, with $l > 1$. Then

$$\begin{aligned}
(a_k \rfloor B_l) \rfloor C_m &= (a_k \rfloor (B_{l-1} \wedge b_l)) \rfloor C_m \\
&= \Big((a_k \rfloor B_{l-1}) \wedge b_l + \widehat{B_{l-1}} \wedge (a_k \rfloor b_l)\Big) \rfloor C_m \\
&= \Big(b_l \wedge (\widehat{a_k \rfloor B_{l-1}})\Big) \rfloor C_m + (a_k \rfloor b_l)\Big(\widehat{B_{l-1}} \rfloor C_m\Big) \\
&= b_l \rfloor \Big(\Big(\widehat{a_k} \rfloor \widehat{B_{l-1}}\Big) \rfloor C_m\Big) + (a_k \rfloor b_l)\Big(\widehat{B_{l-1}} \rfloor C_m\Big) \\
&= b_l \rfloor \Big(\widehat{a_k} \wedge \Big(\widehat{B_{l-1}} \rfloor C_m\Big)\Big) + (a_k \rfloor b_l)\Big(\widehat{B_{l-1}} \rfloor C_m\Big) \\
&= (b_l \rfloor \widehat{a_k}) \wedge \Big(\widehat{B_{l-1}} \rfloor C_m\Big) + a_k \wedge \Big(b_l \rfloor \Big(\widehat{B_{l-1}} \rfloor C_m\Big)\Big) + (a_k \rfloor b_l)\Big(\widehat{B_{l-1}} \rfloor C_m\Big) \\
&= a_k \wedge \Big(b_l \rfloor \Big(\widehat{B_{l-1}} \rfloor C_m\Big)\Big) \\
&= a_k \wedge \Big(\Big(b_l \wedge \widehat{B_{l-1}}\Big) \rfloor C_m\Big) \\
&= a_k \wedge ((B_{l-1} \wedge b_l) \rfloor C_m) \\
&= a_k \wedge (B_l \rfloor C_m).
\end{aligned} \tag{4.10.27}$$

Assume the result holds for $k - 1$. Then

$$\begin{aligned}
A_k \wedge (B_l \rfloor C_m) &= (A_{k-1} \wedge a_k) \wedge (B_l \rfloor C_m) \\
&= a_k \wedge \Big(\widehat{A_{k-1}} \wedge (B_l \rfloor C_m)\Big) \\
&= a_k \wedge \Big(\Big(\widehat{A_{k-1}} \rfloor B_l\Big) \rfloor C_m\Big) \\
&= \Big(a_k \rfloor \Big(\widehat{A_{k-1}} \rfloor B_l\Big)\Big) \rfloor C_m \\
&= \Big(\Big(a_k \wedge \widehat{A_{k-1}}\Big) \rfloor B_l\Big) \rfloor C_m \\
&= ((A_{k-1} \wedge a_k) \rfloor B_l) \rfloor C_m \\
&= (A_k \rfloor B_l) \rfloor C_m.
\end{aligned} \tag{4.10.28}$$

□

Theorem 4.10.14 (Geometric product is sometimes contraction). *Let $A_k, B_l \in \mathcal{C}l(V)$ be a k-blade and an l-blade, respectively. If span$(A_k) \subset$ span(B_l), then*

$$A_k B_l = A_k \rfloor B_l. \tag{4.10.29}$$

Proof. Let $B_l = b_1 \wedge \cdots \wedge b_l$, where $\{b_1, \ldots, b_l\} \subset V$, which we may assume to be orthogonal by Theorem 4.5.8. Since $\text{span}(A_k) \subset \text{span}(B_l)$, $A_k = a_1 \wedge \cdots \wedge a_k$, where

$a_i = \sum_{j=1}^{l} \alpha_{i,j} b_j$, and $\alpha_{i,j} \in \mathbb{R}$. Now

$$\begin{aligned}
A_k B_l &= \left(\left(\sum_{j_1=1}^{l} \alpha_{1,j_1} b_{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_k=1}^{l} \alpha_{k,j_k} b_{j_k} \right) \right) (b_1 \wedge \cdots \wedge b_l) \\
&= \sum_{j_1=1}^{l} \cdots \sum_{j_k=1}^{l} \alpha_{1,j_1} \cdots \alpha_{k,j_k} (b_{j_1} \wedge \cdots \wedge b_{j_k})(b_1 \wedge \cdots \wedge b_l) \\
&= \sum_{|\{j_1,\dots j_k\}|=k} \alpha_{1,j_1} \cdots \alpha_{k,j_k} (b_{j_1} \wedge \cdots \wedge b_{j_k})(b_1 \wedge \cdots \wedge b_l) \\
&= \sum_{|\{j_1,\dots j_k\}|=k} \alpha_{1,j_1} \cdots \alpha_{k,j_k} b_{j_1} \cdots b_{j_k} b_1 \cdots b_l \\
&= \langle A_k B_l \rangle_{l-k} \\
&= A_k \rfloor B_l,
\end{aligned} \tag{4.10.30}$$

where we also used Theorem 4.5.9. □

4.11 Dual

Let $B_l \in \mathcal{C}l(V)$ be an invertible l-blade. The **dual on** B_l is a function ${}^{\Vdash B_l} : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A^{\Vdash B_l} = A \rfloor B_l^{-1}. \tag{4.11.1}$$

If $B_l \in \mathcal{C}l(V)$ is an l-blade, then we will denote

$$\operatorname{span}(A)^{\Vdash B_l} = \operatorname{span}(A)^{\Vdash \operatorname{span}(B_l)}. \tag{4.11.2}$$

A **pseudo-scalar** of a subspace $S \subset V$ is any l-blade $B_l \in \mathcal{C}l(V)$ such that $\operatorname{span}(B_l) = S$.

Theorem 4.11.1 (Dual is linear). *The dual is linear.*

Proof. The contraction is bilinear, and therefore in particular linear in its first argument. □

Remark 4.11.2. If $A_k, B_l \in \mathcal{C}l(V)$ are a k-blade and an invertible l-blade, respectively, and $\operatorname{span}(A_k) \subset \operatorname{span}(B_l)$, then by Theorem 4.10.14

$$A_k^{\Vdash B_l} = A_k B_l^{-1}. \tag{4.11.3}$$

Theorem 4.11.3 (Span of the dual). *Let $A_k, B_l \in \mathcal{C}l(V)$ be an invertible k-blade, and an l-blade, respectively. Then*

$$\mathit{span}(A_k)^{\Vdash B_l} = \{y \in \mathit{span}(B_l) : y \rfloor A_k = 0\}. \tag{4.11.4}$$

Proof. Let $x, y \in V$ such that $x \wedge A_k = 0$, and $y \rfloor A_k = 0$. Then by Theorem 4.8.10,

$$\begin{aligned}
y \rfloor (x \wedge A_k) &= (y \rfloor x) A_k - x(y \rfloor A_k) \\
&= (y \cdot x) A_k \\
&= 0.
\end{aligned} \tag{4.11.5}$$

Since A_k is invertible, $x \cdot y = 0$. If in addition $y \in \operatorname{span}(B_l)$, then $y \in \operatorname{span}(A_k)^{\Vdash B_l}$. □

Theorem 4.11.4 (Span of dual is orthogonal complement of primal span). *If $A_k, B_l \in \mathcal{C}l(V)$ are a k-blade and an invertible l-blade, respectively, then*

$$\mathit{span}\big(A_k{}^{\Vdash B_l}\big) = \mathit{span}(A_k)^{\Vdash B_l}. \tag{4.11.6}$$

Proof. Assume $y \in \operatorname{span}(A_k)^{\Vdash B_l}$. By Theorem 4.11.3, $y \in \operatorname{span}(B_l)$, and $y \rfloor A_k = 0$. Then by Theorem 4.10.13

$$\begin{aligned} y \wedge (A_k \rfloor B_l^{-1}) &= (y \rfloor A_k) \rfloor B_l^{-1} \\ &= 0. \end{aligned} \tag{4.11.7}$$

Therefore $y \in \operatorname{span}\big(A_k{}^{\Vdash B_l}\big)$. Assume $y \in \operatorname{span}\big(A_k{}^{\Vdash B_l}\big)$. Then $y \in \operatorname{span}(B_l)$. By Theorem 4.10.13 and Theorem 4.10.14,

$$\begin{aligned} y \wedge (A_k \rfloor B_l^{-1}) &= (y \rfloor A_k) \rfloor B_l^{-1} \\ &= (y \rfloor A_k) B_l^{-1} \\ &= 0. \end{aligned} \tag{4.11.8}$$

Since B_l is invertible, $y \rfloor A_k = 0$. Therefore $y \in \operatorname{span}\big(A_k{}^{\Vdash B_l}\big)$. □

Theorem 4.11.5 (Inverse of the dual). *If $A_k \in \mathcal{C}l(V)$ is a k-blade, and $B_l \in \mathcal{C}l(V)$ is an invertible l-blade such that $\mathit{span}(A_k) \subset \mathit{span}(B_l)$, then*

$$\big(A_k{}^{\Vdash B_l}\big)^{\Vdash B_l^{-1}} = A_k = \Big(A_k{}^{\Vdash B_l^{-1}}\Big)^{\Vdash B_l}. \tag{4.11.9}$$

Proof. By Theorem 4.10.13, and Theorem 4.9.4,

$$\begin{aligned} \big(A_k{}^{\Vdash B_l}\big)^{\Vdash B_l^{-1}} &= \big(A_k \rfloor B_l^{-1}\big) \rfloor B_l \\ &= A_k \wedge \big(B_l^{-1} \rfloor B_l\big) \\ &= A_k \\ &= A_k \wedge \big(B_l \rfloor B_l^{-1}\big) \\ &= (A_k \rfloor B_l) \rfloor B_l^{-1} \\ &= \Big(A_k{}^{\Vdash B_l^{-1}}\Big)^{\Vdash B_l}. \end{aligned} \tag{4.11.10}$$

□

Example 4.11.6. Let $u, v \in V$ be invertible vectors in $\mathcal{C}l(V)$, with $u \cdot v = 0$, and $B_2 = u \wedge v$. Let $a = \alpha u + \beta v \in \operatorname{span}(B_2) = S$, for some $\alpha, \beta \in \mathbb{R}$. The dual of a on B_2 is given by

$$\begin{aligned} a^{\Vdash B_2} &= a \rfloor B_2^{-1} \\ &= a \rfloor (u \wedge v)^{-1} \\ &= a \rfloor (uv)^{-1} \\ &= a \rfloor (v^{-1}u^{-1}) \\ &= a \rfloor (v^{-1} \wedge u^{-1}) \\ &= (a \cdot v^{-1})u - (a \cdot u^{-1})v \\ &= \beta u - \alpha v, \end{aligned} \tag{4.11.11}$$

where we used Theorem 4.9.3. Now $a^{\Vdash B_2} \in S$, and $a^{\Vdash B_2} \cdot a = 0$. Let $A_2 = \alpha B_2$. The dual of A_2 on B_2 is given by

$$\begin{aligned} {A_2}^{\Vdash B_2} &= A_2 \rfloor B_2^{-1} \\ &= (\alpha B_2) \rfloor B_2^{-1} \\ &= \alpha B_2 \rfloor B_2^{-1} \\ &= \alpha. \end{aligned} \tag{4.11.12}$$

The dual of $\alpha \in \mathbb{R}$ on B_2 is given by

$$\begin{aligned} \alpha^{\Vdash B_2} &= \alpha \rfloor B_2^{-1} \\ &= \alpha B_2^{-1}. \end{aligned} \tag{4.11.13}$$

In general, we conclude that if $A_k \in \mathcal{C}l(V)$ is a k-blade such that $\operatorname{span}(A_k) \subset S$, then ${A_k}^{\Vdash B_2}$ has $\operatorname{span}(A_k)^{\Vdash S} \subset S$, and a well-defined orientation and magnitude.

4.12 Commutator product

The **commutator product** is a function $\times : \mathcal{C}l(V)^2 \to \mathcal{C}l(V)$ defined by

$$A \times B = \frac{1}{2}(AB - BA). \tag{4.12.1}$$

Remark 4.12.1. The commutator product of $A \in \mathcal{C}l(V)$ and $B \in \mathcal{C}l(V)$ is zero if and only if A and B commute with respect to the geometric product. This explains the name.

Theorem 4.12.2 (Commutator product is bilinear). *The commutator product is bilinear.*

Proof. The geometric product is bilinear. □

Theorem 4.12.3 (Commutator product anti-commutes). *Let $A, B \in \mathcal{C}l(V)$. Then*

$$A \times B = -B \times A. \tag{4.12.2}$$

Proof.

$$\begin{aligned} 2(A \times B) &= AB - BA \\ &= -(BA - AB) \\ &= -2(B \times A). \end{aligned} \tag{4.12.3}$$

□

Theorem 4.12.4 (Jacobi identity for the commutator product). *Let $A, B, C \in \mathcal{C}l(V)$. Then*

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0. \tag{4.12.4}$$

Proof.

$$\begin{aligned}4(A \times (B \times C)) &= ABC - ACB - BCA + CBA \\ 4(B \times (C \times A)) &= BCA - BAC - CAB + ACB \\ 4(C \times (A \times B)) &= CAB - CBA - ABC + BAC.\end{aligned} \tag{4.12.5}$$

Adding these equations together proves the claim. □

Remark 4.12.5. The commutator product is not associative, but instead satisfies the Jacobi identity, as given in Theorem 4.12.4.

Theorem 4.12.6 (Commutator product over a geometric product). *Let $A \in \mathcal{C}l(V)$, and $\{B_1, \dots, B_k\} \subset \mathcal{C}l(V)$. Then*

$$\begin{aligned}A \times (B_1 \cdots B_k) &= \sum_{i=1}^{k} B_1 \cdots B_{i-1}(A \times B_i)B_{i+1} \cdots B_k \\ (B_1 \cdots B_k) \times A &= \sum_{i=1}^{k} B_1 \cdots B_{i-1}(B_i \times A)B_{i+1} \cdots B_k.\end{aligned} \tag{4.12.6}$$

Proof. The result holds trivially for $k = 1$. Assume the result holds for $k - 1$, where $k > 1$. Then by induction

$$\begin{aligned}2(A \times (B_1 \cdots B_k)) &= A(B_1 \cdots B_k) - (B_1 \cdots B_k)A \\ &= [A(B_1 \cdots B_{k-1})B_k - (B_1 \cdots B_{k-1})AB_k] + [(B_1 \cdots B_{k-1})AB_k - (B_1 \cdots B_{k-1})B_k A] \\ &= 2(A \times (B_1 \cdots B_{k-1}))B_k + 2(B_1 \cdots B_{k-1})(A \times B_k) \\ &= 2\left[\sum_{i=1}^{k-1} B_1 \cdots B_{i-1}(A \times B_i)B_{i+1} \cdots B_{k-1}B_k\right] + 2(B_1 \cdots B_{k-1})(A \times B_k) \\ &= 2\sum_{i=1}^{k} B_1 \cdots B_{i-1}(A \times B_i)B_{i+1} \cdots B_k.\end{aligned} \tag{4.12.7}$$

The latter claim follows from the skew-symmetry of the commutator product. □

Theorem 4.12.7. *Let $a \in V$, and $B \in \mathcal{C}l(V)$. Then*

$$a \times B = a \rfloor B_+ + a \wedge B_-. \tag{4.12.8}$$

4.13 Orthogonality

Let $B = \{e_1, \dots, e_n\} \subset V$, and $I = \{i_1, \dots, i_k\} \subset [1, n]$, where $0 \leq k \leq n$, and $1 \leq i_1 < \cdots < i_k \leq n$. Then $e_I := e_{i_1} \cdots e_{i_k}$ is called an **ordered k-versor** in B.

Remark 4.13.1. The ordered k-versor in B is my own terminology.

Theorem 4.13.2 (Orthogonal non-null set has linearly independent ordered versors). *If $B = \{b_1, \dots, b_k\} \subset V$ is an orthogonal non-null set, and $k \geq 2$, then the ordered versors in B are linearly independent.*

Proof. First decompose

$$\begin{aligned}\sum_{I\subset[1,k]} \alpha_I b_I &= \sum_{I\subset[1,k-1]} \alpha_I b_I + \sum_{I\subset[1,k-1]} \alpha_{I\cup\{k\}} b_I b_k \\ &= \sum_{I\subset[1,k-1]} \big(\alpha_I + \alpha_{I\cup\{k\}}(-1)^{|I|} b_k\big) b_I.\end{aligned} \tag{4.13.1}$$

Assume

$$\begin{aligned}&\sum_{I\subset[1,k]} \alpha_I b_I = 0 \\ \Leftrightarrow &\sum_{I\subset[1,k-1]} \big(\alpha_I + \alpha_{I\cup\{k\}}(-1)^{|I|} b_n\big) b_I = 0\end{aligned} \tag{4.13.2}$$

Multiplying from the left by $b_n b_i$, and from the right by $\frac{b_i b_n}{b_i^2 b_n^2}$, where $i < k$, is an invertible operation which gives

$$\Leftrightarrow \sum_{I\subset[1,k-1]} \big(-\alpha_I + \alpha_{I\cup\{k\}}(-1)^{|I|} b_n\big) b_I = 0. \tag{4.13.3}$$

Therefore, by the sum and difference of the previous equations, we obtain

$$\begin{aligned}\Leftrightarrow &\sum_{I\subset[1,k-1]} \alpha_I b_I = 0 \text{ and} \\ &\sum_{I\subset[1,k-1]} \alpha_{I\cup\{k\}}(-1)^{|I|} b_n b_I = 0,\end{aligned} \tag{4.13.4}$$

which, by multiplying the second equation with $\frac{b_n}{b_n^2}$ from the right gives

$$\begin{aligned}\Leftrightarrow &\sum_{I\subset[1,k-1]} \alpha_I b_I = 0 \text{ and} \\ &\sum_{I\subset[1,k-1]} \alpha_{I\cup\{k\}} b_I = 0.\end{aligned} \tag{4.13.5}$$

Therefore, $\{b_I\}_{I\subset[1,k]} \subset \mathcal{C}l(V)$ is linearly independent if and only if $\{b_I\}_{I\subset[1,k-1]} \subset \mathcal{C}l(V)$ is linearly independent. Since our proof makes use of both b_i and b_k, the base case is to prove $\{b_\emptyset, b_{\{1\}}\} \subset \mathcal{C}l(V)$ linearly independent. Let

$$\alpha_0 b_0 + \alpha_1 b_1 = 0, \tag{4.13.6}$$

where we used 0 and 1 instead of $\emptyset$ and $\{1\}$. Since $k \geq 2$, b_2 exists. Multiplying Equation 4.13.6 from the left by b_2, and then from the right by b_2, and summing up, we get $\alpha_0 = 0$, which implies $\alpha_1 = 0$. Therefore B is linearly independent. $\square$

Theorem 4.13.3 (Ordered versors of a singleton set may be linearly independent). *If $\{e_1\} \subset V \setminus \{0\}$, and $e_1 \cdot e_1 \leq 0$, then $\{1, e_1\} \subset \mathcal{C}l(V)$ is linearly independent.*

Proof. Let
$$\alpha_0 e_0 + \alpha_1 e_1 = 0. \tag{4.13.7}$$
If $\alpha_0 = 0$, then $\alpha_1 = 0$ by the definition of a vector space, and vice versa; assume both are non-zero. Then $e_1 = -\frac{\alpha_0}{\alpha_1} e_0$, and it follows by squaring that $e_1 \cdot e_1 = \frac{\alpha_0^2}{\alpha_1^2} > 0$. Therefore, if $e_1 \cdot e_1 \leq 0$, this is a contradiction, and the result holds. □

Remark 4.13.4 (Ordered versors of a singleton set may be linearly dependent). If $\{e_1\} \subset V$ is a basis of V, and $e_1 \cdot e_1 > 0$, then $\{1, e_1\} \subset \mathcal{C}l(V)$ needs to be required linearly independent; it can not be proved as such.

Theorem 4.13.5 (Ordered versors of an orthogonal non-null basis of V is a basis of $\mathcal{C}l(V)$). *Let $B \subset V$ be an orthogonal non-null basis of V. Then the ordered versors in B form a basis of $\mathcal{C}l(V)$.*

Proof. The B is linearly independent by Theorem 2.3.3. TODO. □

Theorem 4.13.6 (Clifford algebra can be implemented on a computer). *Clifford algebra $\mathcal{C}l(V)_{p,q,r}$ is algebra-isomorphic to $\mathcal{C}l(\mathbb{R}^{p,q,r})$.*

Proof. Let $\{e_1, \dots e_n\} \subset \mathbb{R}^{p,q,r}$ be the standard basis, where $n = p + q + r$. By Theorem 2.5.2, there exists an orthonormal basis $B = \{b_1, \dots, b_n\} \subset V$ such that the dot product $\cdot$ is of the required form on the vectors of B. Let $f : \mathbb{R}^n \to V$ be a linear function such that $f(e_i) = b_i$ (which is then bijective). Then this function can be extended to an algebra isomorphism $\phi : \mathcal{C}l(V) \to \mathcal{C}l(\mathbb{R}^{p,q,r})$ by requiring linearity, $\phi_{|V} = f$, and $\phi(a_1 \cdots a_k) = \phi(a_1) \cdots \phi(a_k)$. □

Remark 4.13.7. By Theorem 4.13.6, it may seem that $\mathcal{C}l(\mathbb{R}^{p,q,r})$ is all you need, and that everything else is just notation. This is true. However, concentrating on $\mathcal{C}l(\mathbb{R}^{p,q,r})$ brings back not only the temptation for coordinate-dependent proofs (since it has an orthogonal basis for V), but also for proofs which depend on the specific form of the dot product. This has the effect of hiding structure in proofs, something which we want to avoid.

Theorem 4.13.8 (Center of a Clifford algebra). *Let $I_n \in \mathcal{C}l(V)$ be a non-zero n-blade. Then*
$$\mathit{Center}(\mathcal{C}l(V)) = \begin{cases} \mathbb{R}, & \text{if } \dim(V) \text{ is even}, \\ \{\alpha + \beta I_n : \alpha, \beta \in \mathbb{R}\}, & \text{if } \dim(V) \text{ is odd}, \end{cases} \tag{4.13.8}$$

Proof. Let $B = \{b_1, \dots, b_n\} \subset V$ be an orthogonal basis of V, and $A \in \operatorname{Center}(\mathcal{C}l(V))$. Let $I \subset [1, n]$, such that $\emptyset \neq I \neq [1, n]$. Let $i \in I$, and $j \in [1, n] \setminus I$. Then
$$\begin{aligned} b_I b_i b_j &= (-1)^{|I|-1} b_i b_I b_j \\ &= (-1)^{|I|-1}(-1)^{|I|} b_i b_j b_I \\ &= -b_i b_j b_I. \end{aligned} \tag{4.13.9}$$
Therefore $b_I \notin \operatorname{Center}(\mathcal{C}l(V))$. Clearly $b_\emptyset \in \operatorname{Center}(\mathcal{C}l(V))$. Let $J \subset [1, n]$. Then
$$\begin{aligned} b_{[1,n]} b_J &= (-1)^{(|J|-1)|J| + |J|(n-|J|)} b_J b_{[1,n]} \\ &= (-1)^{|J|(n-1)} b_J b_{[1,n]}. \end{aligned} \tag{4.13.10}$$
Thus $b_{[1,n]} \in \operatorname{Center}(\mathcal{C}l(V))$ if and only if n is odd. □

4.14 Magnitude

The **magnitude** in $\mathcal{C}l(V)$ is a function $\|\cdot\| : \mathcal{C}l(V) \to \mathbb{R}$ defined by

$$\|A\| = \sqrt{|A * A|}. \tag{4.14.1}$$

Remark 4.14.1. Sometimes the magnitude is also called the *weight*. However, this choice conflicts with another concept in those models of geometry which incorporate the point at the origin. Following [4], we choose the term magnitude to avoid such confusion.

Theorem 4.14.2 (Norms from bilinear forms). *The $\|\cdot\|$ is a norm (semi-norm) if and only if $\cdot$ is positive-definite (positive-semi-definite).*

Proof. By Theorem 2.9.8 $\|\cdot\|$ is a norm (semi-norm) if and only if the scalar product is a definite (semi-definite) symmetric bilinear form. By Theorem 4.7.8, the scalar product $*$ is positive-definite (positive-semi-definite) if and only if $\cdot$ is positive-definite (positive-semi-definite). □

Remark 4.14.3. In $\mathbb{R}^{p,q}$, if $p = 1$ and $q > 0$, or $p > 0$ and $q = 1$, (e.g. the space-time of special relativity) the corresponding function $\sqrt{|a \cdot a|}$ is sometimes called the Minkowski norm (not to be confused with the L_p-norms), although this function fails the triangle inequality. We use the term magnitude instead to avoid confusion.

Theorem 4.14.4 (Orthogonal functions preserve magnitude). *Let $f : \mathcal{C}l(V) \to \mathcal{C}l(W)$ be orthogonal. Then*

$$\|f(A)\| = \|A\|. \tag{4.14.2}$$

Proof.

$$\begin{aligned} \|f(A)\|^2 &= |f(A) * f(A)| \\ &= |A * A| \\ &= \|A\|^2. \end{aligned} \tag{4.14.3}$$

□

Remark 4.14.5. In particular, Theorem 4.14.4 applies to grade involution, reversion, and conjugation.

4.15 Exponential and trigonometric functions

Let R be a topological associative unital $\mathbb{R}$-algebra. The **exponential**, **sine**, **cosine**, **hyperbolic sine**, and **hyperbolic cosine** are functions $R \to R$ defined by

$$\begin{aligned} e^A &= \sum_{i=0}^{\infty} \frac{A^i}{i!}, \\ \sin(A) &= \sum_{i=0}^{\infty} (-1)^i \frac{A^{2i+1}}{(2i+1)!}, \\ \cos(A) &= \sum_{i=0}^{\infty} (-1)^i \frac{A^{2i}}{(2i)!}, \\ \sinh(A) &= \sum_{i=0}^{\infty} \frac{A^{2i+1}}{(2i+1)!}, \\ \cosh(A) &= \sum_{i=0}^{\infty} \frac{A^{2i}}{(2i)!}. \end{aligned} \tag{4.15.1}$$

Theorem 4.15.1 (Exponential for blades). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade. Then*

$$e^{A_k} = \begin{cases} \cosh(\alpha) + A_k \frac{\sinh(\alpha)}{\alpha}, & \text{if } A_k^2 = \alpha^2 \\ 1 + A_k, & \text{if } A_k^2 = 0 \\ \cos(\alpha) + A_k \frac{\sin(\alpha)}{\alpha}, & \text{if } A_k^2 = -\alpha^2. \end{cases} \tag{4.15.2}$$

Proof. Assume $A_k^2 = \alpha^2$. Then

$$\begin{aligned} e^{A_k} &= \sum_{i=0}^{\infty} \frac{A_k^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{A_k^{2i}}{(2i)!} + \sum_{i=0}^{\infty} \frac{A_k^{2i+1}}{(2i+1)!} \\ &= \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i)!} + A_k \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i+1)!} \\ &= \cosh(\alpha) + A_k \frac{\sinh(\alpha)}{\alpha} \end{aligned} \tag{4.15.3}$$

Assume $A_k^2 = 0$. Then

$$\begin{aligned} e^{A_k} &= \sum_{i=0}^{\infty} \frac{A_k^i}{i!} \\ &= 1 + A. \end{aligned} \tag{4.15.4}$$

Assume $A_k^2 = -\alpha^2$. Then

$$\begin{aligned} e^{A_k} &= \sum_{i=0}^{\infty} \frac{A_k^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{A_k^{2i}}{(2i)!} + \sum_{i=0}^{\infty} \frac{A_k^{2i+1}}{(2i+1)!} \\ &= \sum_{i=0}^{\infty} (-1)^i \frac{\alpha^{2i}}{(2i)!} + A_k \sum_{i=0}^{\infty} (-1)^i \frac{\alpha^{2i}}{(2i+1)!} \\ &= \cos(\alpha) + A_k \frac{\sin(\alpha)}{\alpha} \end{aligned} \tag{4.15.5}$$

□

Theorem 4.15.2 (Sine for blades). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade, $\alpha \in \mathbb{R}$, and $\alpha \neq 0$. Then*

$$\sin(A_k) = \begin{cases} A_k \frac{\sin(\alpha)}{\alpha}, & \text{if } A_k^2 = \alpha^2 \\ A_k, & \text{if } A_k^2 = 0 \\ A_k \frac{\sinh(\alpha)}{\alpha}, & \text{if } A_k^2 = -\alpha^2. \end{cases} \tag{4.15.6}$$

Proof. Assume $A_k^2 = \alpha^2$. Then

$$\begin{aligned} \sin(A_k) &= \sum_{i=0}^{\infty} (-1)^i \frac{A_k^{2i+1}}{(2i+1)!} \\ &= A_k \sum_{i=0}^{\infty} (-1)^i \frac{\alpha^{2i}}{(2i+1)!} \\ &= A_k \frac{\sin(\alpha)}{\alpha}. \end{aligned} \tag{4.15.7}$$

Assume $A_k^2 = 0$. Then

$$\begin{aligned} \sin(A_k) &= \sum_{i=0}^{\infty} (-1)^i \frac{A_k^{2i+1}}{(2i+1)!} \\ &= A_k. \end{aligned} \tag{4.15.8}$$

Assume $A_k^2 = -\alpha^2$. Then

$$\begin{aligned} \sin(A_k) &= \sum_{i=0}^{\infty} (-1)^i \frac{A_k^{2i+1}}{(2i+1)!} \\ &= A_k \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i+1)!} \\ &= A_k \frac{\sinh(\alpha)}{\alpha}. \end{aligned} \tag{4.15.9}$$

□

Theorem 4.15.3 (Cosine for blades). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade, $\alpha \in \mathbb{R}$, and $\alpha \neq 0$. Then*

$$\cos(A_k) = \begin{cases} \cos(\alpha), & \text{if } A_k^2 = \alpha^2 \\ 1, & \text{if } A_k^2 = 0 \\ \cosh(\alpha), & \text{if } A_k^2 = -\alpha^2. \end{cases} \tag{4.15.10}$$

Proof. Assume $A_k = \alpha^2$. Then

$$\begin{aligned} \cos(A_k) &= \sum_{i=0}^{\infty} (-1)^i \frac{A_k^{2i}}{(2i)!} \\ &= \sum_{i=0}^{\infty} (-1)^i \frac{\alpha^{2i}}{(2i)!} \\ &= \cos(\alpha). \end{aligned} \tag{4.15.11}$$

Assume $A_k = 0$. Then

$$\begin{aligned} \cos(A_k) &= \sum_{i=0}^{\infty} (-1)^i \frac{A_k^{2i}}{(2i)!} \\ &= 1. \end{aligned} \tag{4.15.12}$$

Assume $A_k = -\alpha^2$. Then

$$\begin{aligned} \cos(A_k) &= \sum_{i=0}^{\infty} (-1)^i \frac{A_k^{2i}}{(2i)!} \\ &= \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i)!} \\ &= \cosh(\alpha). \end{aligned} \tag{4.15.13}$$

□

Theorem 4.15.4 (Hyperbolic cosine for blades). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade. Then*

$$\cosh(A_k) = \begin{cases} \cosh(\alpha), & \text{if } A_k^2 = \alpha^2 \\ 1, & \text{if } A_k^2 = 0 \\ \cos(\alpha), & \text{if } A_k^2 = -\alpha^2. \end{cases} \tag{4.15.14}$$

Proof. Similarly as in the proof for the cosine. □

Theorem 4.15.5 (Hyperbolic sine for blades). *Let $A_k \in \mathcal{C}l(V)$ be a k-blade. Then*

$$\sinh(A_k) = \begin{cases} A_k \frac{\sinh(\alpha)}{\alpha}, & \text{if } A_k^2 = \alpha^2 \\ A_k, & \text{if } A_k^2 = 0 \\ A_k \frac{\sin(\alpha)}{\alpha}, & \text{if } A_k^2 = -\alpha^2. \end{cases} \tag{4.15.15}$$

Proof. Similarly as in the proof for the sine. □

5 Linear transformations

In this section we study the linear transformations of V, extended as outermorphisms to all of $\mathcal{C}l(V)$. The orthogonal outermorphisms in $\mathcal{C}l(V)$ have a particularly powerful representation as pinors.

5.1 Determinant

The **determinant** is a function $\det : \mathrm{Out}(V) \to \mathbb{R}$ defined by

$$\det(f) = f(A_n)A_n^{-1}, \tag{5.1.1}$$

for any invertible n-blade $A_n \in \mathcal{C}l(V)$, where $n = \dim(V)$.

Theorem 5.1.1 (Determinant is well-defined). *Let $f \in Out(V)$, and $A_n, B_n \in \mathcal{C}l(V)$ be invertible n-blades. Then*

$$f(A_n)A_n^{-1} = f(B_n)B_n^{-1}. \tag{5.1.2}$$

Proof. First, notice that

$$A_n B_n^{-1} = A_n \rfloor B_n^{-1} \in \mathbb{R}. \tag{5.1.3}$$

Then

$$\begin{aligned} f(A_n)A_n^{-1} &= f(A_n B_n^{-1} B_n)(A_n B_n^{-1} B_n)^{-1} \\ &= (A_n B_n^{-1})\big[f(B_n)B_n^{-1}\big](A_n B_n^{-1})^{-1} \\ &= f(B_n)B_n^{-1}. \end{aligned} \tag{5.1.4}$$

□

Remark 5.1.2 (Determinant is not additive). If $f, g \in \mathrm{Out}(V)$, then $f+g \notin \mathrm{Out}(V)$ by Theorem 2.7.5, and so it may not be deduced that $\det(f+g) = \det(f) + \det(g)$, since the left term is not defined.

Theorem 5.1.3 (Determinant of a scaled function). *Let $f \in Out(V)$, and $\alpha \in \mathbb{R}$. Then*

$$det(\alpha f) = \alpha^n det(f). \tag{5.1.5}$$

Proof. Let $A_n = a_1 \wedge \cdots \wedge a_n \in \mathcal{C}l(V)$, where $\{a_1, \dots, a_n\} \subset V$. Then

$$\begin{aligned} \det(\alpha f) &= (\alpha f)(A_n)A_n^{-1} \\ &= (\alpha f)(a_1 \wedge \cdots \wedge a_n)A_n^{-1} \\ &= [(\alpha f)(a_1) \wedge \cdots \wedge (\alpha f)(a_n)]A_n^{-1} \\ &= [(\alpha f(a_1)) \wedge \cdots \wedge (\alpha f(a_n))]A_n^{-1} \\ &= \alpha^n [f(a_1) \wedge \cdots \wedge f(a_n)]A_n^{-1} \\ &= \alpha^n f(a_1 \wedge \cdots \wedge a_n)A_n^{-1} \\ &= \alpha^n f(A_n)A_n^{-1} \\ &= \alpha^n \det(f). \end{aligned} \tag{5.1.6}$$

□

Theorem 5.1.4 (Determinant of the composition). *Let $f, g \in Out(V)$. Then*

$$det(f \circ g) = det(f)\, det(g). \tag{5.1.7}$$

Proof.

$$\begin{aligned}
\det(f \circ g) &= (f \circ g)(A_n)A_n^{-1} \\
&= f(g(A_n))A_n^{-1} \\
&= f\big(g(A_n)A_n^{-1}A_n\big)A_n^{-1} \\
&= f(\det(g)A_n)A_n^{-1} \\
&= \big[f(A_n)A_n^{-1}\big]\det(g) \\
&= \det(f)\det(g).
\end{aligned} \tag{5.1.8}$$

□

5.2 Pinors and unit-versors

A **pinor** is a an invertible element $C \in \mathcal{C}l(V)$ such that $\widehat{C}vC^{-1} \in V$, for all $v \in V$, and $\|C\| = 1$. The **pinor group**, and the **unit-versor group** are groups defined by

$$\begin{aligned}
\mathbf{Pin}(V) &= \{C \in \mathcal{C}l(V) : C \text{ is a pinor}\}, \\
\mathbf{Versor}_1(V) &= \{C \in \mathcal{C}l(V) : C \text{ is a versor}, \|C\| = 1\},
\end{aligned} \tag{5.2.1}$$

respectively, with the geometric product as the group operation. The **pinor transform** along $C \in \mathbf{Pin}(V)$ is an algebra automorphism $\underline{C} : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$\underline{C}(v) = \widehat{C}vC^{-1}, \tag{5.2.2}$$

for all $v \in V$. The **pinor transform** through $C \in \mathbf{Pin}(V)$ is an algebra automorphism $\underline{\underline{C}} : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$\underline{\underline{C}}(v) = \widehat{C}\widehat{v}C^{-1}, \tag{5.2.3}$$

for all $v \in V$.

Remark 5.2.1 (Along and through). The terms along and through become understandable when we consider reflections later.

Remark 5.2.2 (Versor group). The unit-versor group and the pinor group can be extended to a larger group by removing the restriction $\|C\| = 1$. The result is the **versor group**, also known as the **Clifford group**, or the **Lipschitz group**. However, since scaling a pinor by a non-zero real number does not change the induced pinor transform, the class of transformations is not extended.

Theorem 5.2.3 (Pinor transforms are orthogonal). *Let $C \in \mathbf{Pin}(V)$. Then $\underline{C}|V \in \mathbf{O}(V)$, and $\underline{\underline{C}}|V \in \mathbf{O}(V)$.*

Proof. Since $\underline{C}$ is a vector-preserving algebra homomorphism in $\mathcal{C}l(V)$, it is orthogonal by Theorem 4.7.9. Therefore $\underline{C}|V \in \mathbf{O}(V)$. Similarly for $\underline{\underline{C}}|V$. □

Theorem 5.2.4 (Pinor transforms preserve blades). *Let $C \in \mathbf{Pin}(V)$, and $A_k \in \mathcal{C}l(V)$ be a k-blade. Then $\underline{C}(A_k)$ and $\underline{\underline{C}}(A_k)$ are k-blades.*

Proof. Let $A_k = a_1 \cdots a_k \in \mathcal{C}l(V)$, where $\{a_1, \dots, a_k\} \subset V$ is orthogonal. Since $\underline{C}$ is orthogonal by Theorem 5.2.3,

$$\underline{C}(a_i) \cdot \underline{C}(a_j) = a_i \cdot a_j. \tag{5.2.4}$$

Therefore $\underline{C}(A_k)$ is a k-blade. Similarly for $\underline{\underline{C}}(A_k)$. □

Theorem 5.2.5 (Pinor transforms preserve grades). *Let $C \in \mathbf{Pin}(V)$, and $A \in \mathcal{C}l(V)$. Then*

$$\begin{aligned}\underline{C}(\langle A \rangle_k) &= \langle \underline{C}(A) \rangle_k, \\ \underline{\underline{C}}(\langle A \rangle_k) &= \left\langle \underline{\underline{C}}(A) \right\rangle_k.\end{aligned} \tag{5.2.5}$$

Proof. Since the pinor transform along a pinor and grade selection are both linear, we may assume that A is a k-blade. Then $\underline{C}(A) = \underline{C}(\langle A \rangle_k)$ is a k-blade by Theorem 5.2.4. Therefore the pinor transform along a pinor preserves the grade. Similarly for the pinor transform through a pinor. □

Theorem 5.2.6 (Pinor transforms preserves all products). *Let $C \in \mathbf{Pin}(V)$, and $A, B \in \mathcal{C}l(V)$. Then*

$$\begin{aligned}\underline{C}(AB) &= \underline{C}(A)\underline{C}(B), \\ \underline{C}(A \wedge B) &= \underline{C}(A) \wedge \underline{C}(B), \\ \underline{C}(A \rfloor B) &= \underline{C}(A) \rfloor \underline{C}(B), \\ \underline{C}(A * B) &= \underline{C}(A) * \underline{C}(B),\end{aligned} \tag{5.2.6}$$

and similarly for $\underline{\underline{C}}$.

Proof. The geometric product is preserved by the definition of the pinor transforms. The rest of the products are defined in terms of the geometric product, linear combinations, and grade selection. Since linear combinations by definition, and grades are preserved by Theorem 5.2.5, these products also preserved. □

Theorem 5.2.7 (Pinor homomorphism). *Let $\phi : \mathbf{Pin}(V) \to \mathbf{O}(V) : \phi(C) = \underline{C}|V$. Then ϕ is a group homomorphism.*

Proof. Let $C, D \in \mathbf{Pin}(V)$, and $v \in V$. By the associativity of the geometric product,

$$\begin{aligned}\phi(CD)(v) &= \underline{CD}(v) \\ &= \widehat{CD} v (CD)^{-1} \\ &= \widehat{C}\left(\widehat{D} v D^{-1}\right) C^{-1} \\ &= \underline{C}\left(\widehat{D} v D^{-1}\right) \\ &= \underline{C}(\underline{D}(v)) \\ &= [\underline{C} \circ \underline{D}](v) \\ &= [\phi(C) \circ \phi(D)](v).\end{aligned} \tag{5.2.7}$$

□

Theorem 5.2.8 (Pinor non-homomorphism). *Let* $\psi : \mathbf{Pin}(V) \to \mathbf{O}(V) : \psi(C) = \underline{\underline{C}}|V$. *Then* ψ *is not a homomorphism.*

Proof. Let $v \in V$. Then $\psi(1)(v) = -v$, i.e. ψ does not map the identity element in $\mathbf{Pin}(V)$ to the element in $\mathbf{O}(V)$. Therefore ψ is not a homomorphism. $\square$

Theorem 5.2.9 (Kernel of the pinor homomorphism). *Let* $\phi : \mathbf{Pin}(V) \to \mathbf{O}(V) : \phi(C) = \underline{C}|V$. *Then*

$$\phi^{-1}(id_V) = \{-1, +1\}, \tag{5.2.8}$$

where $B_n \in \mathcal{C}l(V)$ *is any unit* n*-blade.*

Proof. Let $v \in V$ and $C \in \mathbf{Pin}(V)$. Then

$$\begin{aligned} & \underline{C}(v) && = v \\ \Leftrightarrow \quad & \widehat{C}vC^{-1} && = v \\ \Leftrightarrow \quad & \widehat{C}v && = vC \\ \Leftrightarrow \quad & \tfrac{vC - \widehat{C}v}{2} && = 0 \\ \Leftrightarrow \quad & v \rfloor C && = 0, \end{aligned}$$

and this holds for all $v \in V$. By linearity of the contraction, and by Theorem 4.8.9, it then holds that

$$A \rfloor C = \langle A \rangle_0 \rfloor C, \tag{5.2.9}$$

for all $A \in \mathcal{C}l(V)$. In particular,

$$\begin{aligned} 1 &= C^{-1} \rfloor C \\ &= \left\langle C^{-1} \right\rangle_0 \rfloor C \\ &= \left\langle C^{-1} \right\rangle_0 C, \end{aligned} \tag{5.2.10}$$

and therefore $C = \langle C^{-1} \rangle_0^{-1} \in \mathbb{R} \setminus \{0\}$. It follows from $\|C\| = 1$ that $C \in \{-1, +1\}$. $\square$

Theorem 5.2.10 (Unit-versor epimorphism). *Let* $\phi : \mathbf{Versor}_1(V) \to \mathbf{O}(V) : \phi(C) = \underline{C}|V$, *Then* ϕ *is a group epimorphism.*

Proof. Let $C = c_1 \cdots c_k$ be a versor, where $\{c_1, \dots, c_k\} \subset V$ are invertible, and $\|C\| = 1$. Then $C \in \mathbf{Pin}(V)$ by Theorem 5.2.13, and $\underline{C}|V \in \mathbf{O}(V)$ by Theorem 5.2.3. The ϕ is a group homomorphism, since it is a sub-group-restriction of a group homomorphism by Theorem 5.2.7. The proof of Theorem 5.2.13 shows that the pinor transform along C is repeated reflection along lines $\{c_1, \dots, c_k\}$ (in that order). Then for every $f \in \mathbf{O}(V)$ there exists a versor $C \in \mathbf{Pin}(V)$ such that $\underline{C} = f$ by Theorem 3.3.6. Therefore ϕ is surjective. $\square$

Theorem 5.2.11 (Pinor homomorphism is an epimorphism). *Let* $\phi : \mathbf{Pin}(V) \to \mathbf{O}(V) : \phi(C) = \underline{C}|V$. *Then* ϕ *is a group epimorphism.*

Proof. The ϕ is a homomorphism by Theorem 5.2.7. Since $\mathbf{Versor}_1(V) \subset \mathbf{Pin}(V)$, and $\mathbf{Versor}_1(V)$ is epimorphic to $\mathbf{O}(V)$ by Theorem 5.2.10, ϕ is an epimorphism. $\square$

Theorem 5.2.12 (Unit vectors are pinors). *Let $c \in V$ be a vector such that $\|c\| = 1$. Then $c \in \mathbf{Pin}(V)$, and*

$$\begin{aligned} \underline{c}(v) &= v - 2(c \cdot v)c, \\ \underline{\underline{c}}(v) &= 2(c \cdot v)c - v, \end{aligned} \tag{5.2.11}$$

for all $v \in V$.

Proof. Since $\|c\| = 1$, c is invertible. Then

$$\begin{aligned} \widehat{c}vc^{-1} &= -(cv)c^{-1} \\ &= -(2(c \cdot v) - vc)c^{-1} \\ &= v - 2(c \cdot v)c \in V, \end{aligned} \tag{5.2.12}$$

and $cvc^{-1} = -\widehat{c}vc^{-1}$. □

Theorem 5.2.13 (Unit versors are pinors).

$$\mathbf{Versor}_1(V) \subset \mathbf{Pin}(V) \tag{5.2.13}$$

Proof. Let $v \in V$, and $C_k = c_1 \cdots c_k$, where $\{c_1, \dots, c_k\} \subset V$ are invertible, and $\|C_k\| = 1$. Then

$$\begin{aligned} \widehat{C_k}vC_k^{-1} &= \widehat{C_{k-1}c_k}v(C_{k-1}c_k)^{-1} \\ &= \widehat{C_{k-1}}\big(\widehat{c_k}vc_k^{-1}\big)C_{k-1}^{-1}. \end{aligned} \tag{5.2.14}$$

Since $\widehat{c_k}vc_k^{-1} \in V$ by Theorem 5.2.12, $\widehat{C_k}vC_k^{-1} \in V$ by induction. Therefore $C_k \in \mathbf{Pin}(V)$. □

Theorem 5.2.14 (Kernel of the unit-versor epimorphism). *Let $\phi : \mathbf{Versor}_1(V) \to \mathbf{O}(V) : \phi(C) = \underline{C}|V$. Then*

$$\phi^{-1}(id_V) = \{-1, +1\}. \tag{5.2.15}$$

Proof. Since $\mathbf{Versor}_1(V) \subset \mathbf{Pin}(V)$ by Theorem 5.2.13, it follows that $\phi^{-1}(\mathrm{id}_V) \subset \{-1, +1\}$ by Theorem 5.2.9. Since -1 and $+1$ are both 0-versors, $\phi^{-1}(\mathrm{id}_V) = \{-1, +1\}$. □

Theorem 5.2.15 (Pinors are unit versors).

$$\mathbf{Pin}(V) = \mathbf{Versor}_1(V). \tag{5.2.16}$$

Proof. Let $\phi : \mathbf{Pin}(V) \to \mathbf{O}(V) : \phi(C) = \underline{C}|V$. Then ϕ is an epimorphism by Theorem 5.2.11. Since $\mathbf{Versor}_1(V) \subset \mathbf{Pin}(V)$ by Theorem 5.2.13, $\phi|\mathbf{Versor}_1(V)$ is an epimorphism by Theorem 5.2.10. In addition, $\phi^{-1}(1) = \{-1, +1\} = (\phi|\mathbf{Versor}_1(V))^{-1}(1)$ by Theorem 5.2.14 and Theorem 5.2.9. It then follows that $\mathbf{Versor}_1(V) = \mathbf{Pin}(V)$ by Theorem 2.1.3. □

Remark 5.2.16. Pinors are either even or odd by Theorem 4.4.9.

Theorem 5.2.17 (Pinor transforms for an odd pinor). *If $C \in \langle \mathbf{Pin}(V) \rangle_-$, then*

$$\begin{aligned} \underline{C}(A) &= C\widehat{A}C^{-1}, \\ \underline{\underline{C}}(A) &= CAC^{-1}, \end{aligned} \tag{5.2.17}$$

for all $A \in \mathcal{C}l(V)$.

Proof. By linearity, we only need to prove the result for k-blades. Let $A_k = a_1 \cdots a_k \in \mathcal{C}l(V)$, where $\{a_1, \dots, a_k\} \subset V$ is orthogonal. Then

$$\begin{aligned} \underline{C}(A_k) &= \underline{C}(a_1 \cdots a_k) \\ &= \underline{C}(a_1) \cdots \underline{C}(a_k) \\ &= \left(\widehat{C} a_1 C^{-1}\right) \cdots \left(\widehat{C} a_k C^{-1}\right) \\ &= \left(-C a_1 C^{-1}\right) \cdots \left(-C a_k C^{-1}\right) \\ &= (-1)^k C a_1 \cdots a_k C^{-1} \\ &= C\widehat{A_k}C^{-1}. \end{aligned} \tag{5.2.18}$$

Similarly for $\underline{\underline{C}}$. $\square$

Theorem 5.2.18 (Pinor transforms for an even pinor). *If $C \in \langle \mathbf{Pin}(V) \rangle_+$, then*

$$\begin{aligned} \underline{C}(A) &= CAC^{-1}, \\ \underline{\underline{C}}(A) &= C\widehat{A}C^{-1}, \end{aligned} \tag{5.2.19}$$

for all $A \in \mathcal{C}l(V)$.

Proof. The proof is similar to that of Theorem 5.2.17. $\square$

Theorem 5.2.19 (Determinants of pinor transforms). *Let $C \in \mathbf{Pin}(V)$. Then*

$$\begin{aligned} det(\underline{C}|V) &= \begin{cases} +1, & \text{if } C \in \langle \mathbf{Pin}(V) \rangle_+, \\ -1, & \text{if } C \in \langle \mathbf{Pin}(V) \rangle_-. \end{cases} \\ det\left(\underline{\underline{C}}|V\right) &= \begin{cases} +1, & \text{if } C \in \langle \mathbf{Pin}(V) \rangle_+, \\ +1, & \text{if } C \in \langle \mathbf{Pin}(V) \rangle_-, \text{ and } dim(V) \text{ is odd}, \\ -1, & \text{if } C \in \langle \mathbf{Pin}(V) \rangle_-, \text{ and } dim(V) \text{ is even}. \end{cases} \end{aligned} \tag{5.2.20}$$

Proof. We may write $C = c_1 \cdots c_k$, where $\{c_1, \dots, c_k\} \subset V$ are invertible unit vectors, by Theorem 5.2.15. Then

$$\begin{aligned} \det(\underline{C}|V) &= \det\left(\underline{c_1 \cdots c_k}|V\right) \\ &= \det\left(\underline{c_1}|V \circ \cdots \circ \underline{c_k}|V\right) \\ &= \det\left(\underline{c_1}|V\right) \cdots \det\left(\underline{c_k}|V\right) \\ &= (-1)^k. \end{aligned} \tag{5.2.21}$$

Let $B_n \in \mathcal{C}l(V)$ be any invertible n-blade, where $n = \dim(V)$. Then

$$\begin{aligned}
\det\left(\underline{\underline{C}}|V\right) &= \det\left(\underline{\underline{C}}\right) \\
&= \underline{\underline{C}}(B_n)B_n^{-1} \\
&= \left(CB_nC^{-1}\right)B_n^{-1} \\
&= (-1)^k\widehat{C}\left(B_nC^{-1}B_n^{-1}\right) \\
&= (-1)^k(-1)^{kn}\widehat{C}\widehat{C^{-1}} \\
&= (-1)^{k(n+1)},
\end{aligned} \tag{5.2.22}$$

□

Theorem 5.2.20 (Pinors have a unit pinor norm). *Let $C \in \mathbf{Pin}(V)$. Then*

$$\begin{aligned}
\widetilde{C}C = C\widetilde{C} \in \{-1, +1\}, \\
\overline{C}C = C\overline{C} \in \{-1, +1\}.
\end{aligned} \tag{5.2.23}$$

Proof. By conjugation,

$$\begin{aligned}
\overline{\widehat{C}vC^{-1}} &= \overline{C^{-1}}\overline{v}\overline{\widehat{C}} \\
&= -\overline{C}^{-1}v\widetilde{C} \\
&= -\widehat{C}vC^{-1},
\end{aligned} \tag{5.2.24}$$

where the last step is because $\widehat{C}vC^{-1} \in V$. Now

$$\begin{aligned}
v &= \overline{C}\widehat{C}vC^{-1}\widetilde{C}^{-1} \\
&= \widehat{\widetilde{C}C}v\left(\widetilde{C}C\right)^{-1}.
\end{aligned} \tag{5.2.25}$$

Therefore $\widetilde{C}C \in \{-1, +1\}$ by Theorem 5.2.9, and $C\widetilde{C} = \widetilde{C\widetilde{C}} = \widetilde{C}C$. Similarly, but using reversion instead, one shows that $\overline{C}C = C\overline{C} \in \{-1, +1\}$. □

Theorem 5.2.21 (Scalar product preserves the geometric product of pinors). *Let $A, B \in \mathbf{Pin}(V)$. Then*

$$(AB) * (AB) = (A * A)(B * B). \tag{5.2.26}$$

Proof.

$$\begin{aligned}
(AB) * (AB) &= \left\langle AB\widetilde{AB}\right\rangle_0 \\
&= \left\langle AB\widetilde{B}\widetilde{A}\right\rangle_0 \\
&= \left(A\widetilde{A}\right)\left(B\widetilde{B}\right) \\
&= \left\langle A\widetilde{A}\right\rangle_0\left\langle B\widetilde{B}\right\rangle_0 \\
&= (A * A)(B * B),
\end{aligned} \tag{5.2.27}$$

where we used Theorem 4.4.13, and Theorem 5.2.20. □

5.3 Spinors and rotors

The **spinor group**, and the **rotor group**, are sub-groups of the pinor group defined by

$$\begin{aligned} \mathbf{Spin}(V) &= \langle \mathbf{Pin}(V) \rangle_+, \\ \mathbf{Rotor}(V) &= \{C \in \mathbf{Spin}(V) : C * C = 1\}, \end{aligned} \tag{5.3.1}$$

respectively. The corresponding elements are called a **spinor**, and a **rotor**, respectively.

Remark 5.3.1 (Naming for pinors). Spinors were named first. According to [1], the name for pinors was originally a joke due to Jean-Pierre Serre. It stuck into use.

Remark 5.3.2 (Rotors are special spinors). A rotor is also called a **special spinor**. In [4], page 106, it is erroneously claimed that not all special spinors are versors. This contradicts Theorem 5.2.15. The error can be traced to the sentence

> From previous considerations, we know that such an equation can obtain for all x only if λ has a scalar and possibly a pseudoscalar part only.

However, there can be no pseudo-scalar part by Theorem 5.2.9.

Remark 5.3.3 (Rotors are rotations). Rotors are the correct generalization of rotations from definite bilinear spaces to indefinite bilinear spaces.

Theorem 5.3.4 (Spinor transforms are special orthogonal). *Let* $C \in \mathbf{Spin}(V)$. *Then* $\underline{C}|V \in \mathbf{SO}(V)$ *and* $\underline{\underline{C}}|V \in \mathbf{SO}(V)$.

Proof. It holds that $\underline{C}|V \in \mathbf{O}(V)$ by Theorem 5.2.3. In addition, $\det(\underline{C}|V) = 1$ by Theorem 5.2.19. Therefore $\underline{C}|V \in \mathbf{SO}(V)$. Similarly for $\underline{\underline{C}}|V$. □

Theorem 5.3.5 (The spinor group is epimorphic to $\mathbf{SO}(V)$). *Let* $\phi : \mathbf{Spin}(V) \to \mathbf{SO}(V) : \phi(C) = \underline{C}|V$. *Then* ϕ *is a group epimorphism.*

Proof. Since $\mathbf{Spin}(V) \subset \mathbf{Pin}(V)$, ϕ is a group homomorphism by Theorem 5.2.11. Since $\langle \mathbf{Versor}_1(V) \rangle_+ \subset \mathbf{Spin}(V)$, ϕ is surjective by Theorem 3.3.6. □

Theorem 5.3.6 (Rotor product is proper special orthogonal). *Let* $C \in \mathbf{Rotor}(V)$. *Then* $\underline{C}|V \in \mathbf{SO}^+(V)$.

Proof. It holds that $\underline{C}|V \in \mathbf{SO}(V)$ by Theorem 5.3.4. TODO. □

Theorem 5.3.7 (The rotor group is epimorphic to $\mathbf{SO}^+(V)$). *Let* $\phi : \mathbf{Spin}(V) \to \mathbf{SO}^+(V) : \phi(C) = \underline{C}|V$. *Then* ϕ *is a group epimorphism.*

Proof. TODO. □

5.4 Projections

Let $B_l \in \mathcal{C}l(V)$ be an invertible l-blade. The **projection** on B_l is a function ${}^{\parallel B_l} : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A^{\parallel B_l} = \left(A \rfloor B_l^{-1}\right) \rfloor B_l. \tag{5.4.1}$$

The **rejection** from B_l is a function ${}^{\perp B_l} : \mathcal{C}l(V) \to \mathcal{C}l(V)$ defined by

$$A^{\perp B_l} = A - A^{\parallel B_l}. \tag{5.4.2}$$

Remark 5.4.1. Projection on and rejection from a blade $B_l \in \mathcal{C}l(V)$ are independent of non-zero scaling of B_l. Only the $\operatorname{span}(B_l)$ matters.

Remark 5.4.2 (Intuition for the projection). The projection can also be written as $A^{\Vdash B_l \Vdash B_l^{-1}}$. This gives an intuition for the projection.

Theorem 5.4.3 (Projection on a blade is an outermorphism). *Let $B_l \in \mathcal{C}l(V)$ be an invertible l-blade. Then the projection on B_l is an outermorphism.*

Proof. By linearity, we only need to prove the result when A is a k-blade. Let $A_k = a_1 \wedge \cdots \wedge a_k \in \mathcal{C}l(V)$, where $\{a_1, \dots, a_k\} \subset V$. Then

$$\begin{aligned}
{A_{k-1}}^{\parallel B_l} \wedge {a_k}^{\parallel B_l} &= \left[\left(A_{k-1} \rfloor B_l^{-1}\right) \rfloor B_l\right] \wedge \left[\left(a_k \rfloor B_l^{-1}\right) \rfloor B_l\right] \\
&= \left\{\left[\left(A_{k-1} \rfloor B_l^{-1}\right) \rfloor B_l\right] \rfloor \left[a_k \rfloor B_l^{-1}\right]\right\} \rfloor B_l \\
&= \left\{\left(\left[\left(A_{k-1} \rfloor B_l^{-1}\right) \rfloor B_l\right] \wedge a_k\right) \rfloor B_l^{-1}\right\} \rfloor B_l \\
&= \left\{\left(a_k \wedge \left[\left(\widehat{A_{k-1}} \rfloor B_l^{-1}\right) \rfloor B_l\right]\right) \rfloor B_l^{-1}\right\} \rfloor B_l \\
&= \left\{a_k \rfloor \left(\left[\left(\widehat{A_{k-1}} \rfloor B_l^{-1}\right) \rfloor B_l\right] \rfloor B_l^{-1}\right)\right\} \rfloor B_l \\
&= \left\{a_k \rfloor \left(\left(\widehat{A_{k-1}} \rfloor B_l^{-1}\right) \wedge \left[B_l \rfloor B_l^{-1}\right]\right)\right\} \rfloor B_l \\
&= \left\{a_k \rfloor \left(\widehat{A_{k-1}} \rfloor B_l^{-1}\right)\right\} \rfloor B_l \\
&= \left\{\left(a_k \wedge \widehat{A_{k-1}}\right) \rfloor B_l^{-1}\right\} \rfloor B_l \\
&= \left(A_k \rfloor B_l^{-1}\right) \rfloor B_l \\
&= {A_k}^{\parallel B_l},
\end{aligned} \tag{5.4.3}$$

where we used Theorem 4.8.9 and Theorem 4.10.13. □

5.5 Reflections

A **reflection along** an invertible l-blade $B_l \in \mathcal{C}l(V)$ is the pinor transform along $B_l / \|B_l\|$. A **reflection through** an invertible l-blade $B_l \in \mathcal{C}l(V)$ is the pinor transform through $B_l / \|B_l\|$.

Theorem 5.5.1 (Reflection along a blade for vectors). *Let $a \in V$, and $B_l \in \mathcal{C}l(V)$ be an invertible unit l-blade. Then*

$$\underline{B_l}(a) = -a^{\parallel B_l} + a^{\perp B_l}. \tag{5.5.1}$$

Proof.

$$
\begin{aligned}
\underline{B_l}(a) &= \widehat{B_l} a B_l^{-1} \\
&= \left(\widehat{B_l} \lfloor a + \widehat{B_l} \wedge a\right) B_l^{-1} \\
&= (-a \rfloor B_l + a \wedge B_l) B_l^{-1} \\
&= (-a \rfloor B_l) B_l^{-1} + (a \wedge B_l) B_l^{-1} \\
&= -\left(a \rfloor B_l^{-1}\right) \rfloor B_l + (a \wedge B_l) B_l^{-1} \\
&= -a^{\parallel B_l} + a^{\perp B_l},
\end{aligned}
\tag{5.5.2}
$$

where we used Theorem 4.8.4, and Theorem 4.10.14 □

Example 5.5.2 (Intuition for a reflection along a blade). Consider the reflection of a vector along an invertible blade $B_l \in \mathcal{C}l(V)$ in the following cases. The case $l = 0$ is called a *reflection along a point*; it does not do anything. The case $l = 1$ is called a *reflection along a line*; it negates the single component along the line. The case $l = n - 1$ is called a *reflection along a plane*; it negates the $n - 1$ components along the plane. The case $l = n$ is called a *reflection along a space*; it negates all the n components. In the general case l components along $\operatorname{span}(B_l)$ are negated.

Theorem 5.5.3 (Reflection through a blade for vectors). *Let $a \in V$, and $B_l \in \mathcal{C}l(V)$ be an invertible unit l-blade. Then*

$$
\underline{\underline{B_l}}(a) = a^{\parallel B_l} - a^{\perp B_l}. \tag{5.5.3}
$$

Proof. This is immediate from Theorem 5.5.1. □

Example 5.5.4 (Intuition for a reflection through a blade). Consider the reflection of a vector through an invertible blade $B_l \in \mathcal{C}l(V)$ in the following cases. The case $l = 0$ is called a *reflection through a point*; it negates all the n components orthogonal to the origin. The case $l = 1$ is called a *reflection through a line*; it negates the $n-1$ components orthogonal to the line. The case $l = n - 1$ is called a *reflection through a plane*; it negates the single component orthogonal to the plane. The case $l = n$ is called a *reflection through a space*; it does not do anything. In the general case $n - l$ components in $\operatorname{span}(B_l)^{\Vdash V}$ are negated.

Remark 5.5.5 (Generalized projections). Let $a \in V$, and $B_l \in \mathcal{C}l(V)$ be an invertible unit l-blade. Then the projection on B_l can equivalently be given by

$$
a^{\parallel B_l} = \frac{1}{2}\left(a - \underline{B_l}(a)\right). \tag{5.5.4}
$$

Since this formula also makes sense when $B \in \mathbf{Pin}(V)$, it is natural to ask whether this generalizes projection in some useful way. The answer is unclear to me; the corresponding outermorphism does not seem to have a simple form.

Theorem 5.5.6 (The relation between the reflections). *Let $B_l \in \mathcal{C}l(V)$ be an invertible unit l-blade, and $I_n \in \mathcal{C}l(V)$ an invertible n-blade. Then*

$$
\begin{aligned}
\underline{B_l}(A)^{\Vdash I_n} &= \underline{\underline{B_l^{\Vdash I_n}}}(A), \\
\underline{\underline{B_l}}(A)^{\Vdash I_n} &= \underline{B_l^{\Vdash I_n}}(A).
\end{aligned}
\tag{5.5.5}
$$

Proof.

$$\begin{aligned} \underline{B_l}(A_k)^{\Vdash I_n} &= \left((-1)^{kl} B_l A_k B_l^{-1}\right) \rfloor I_n^{-1} \\ &= (-1)^{kl} B_l A_k B_l^{-1} I_n^{-1} \\ &= (-1)^{kl} B_l \left(I_n^{-1} I_n\right) A_k (I_n B_l)^{-1} \\ &= (-1)^{kl} \left(B_l I_n^{-1}\right) I_n A_k (I_n B_l)^{-1} \end{aligned} \tag{5.5.6}$$

□

5.6 Adjoint

Let $f : \mathcal{C}l(V) \to \mathcal{C}l(V)$ be an outermorphism. The **adjoint** of f is a function $\widetilde{f} : \mathcal{C}l(V) \to \mathcal{C}l(V)$ such that

$$f(A) * B = A * \widetilde{f}(B), \tag{5.6.1}$$

for all $A, B \in \mathcal{C}l(V)$.

Remark 5.6.1. Since $*$ is non-degenerate, the adjoint is well-defined.

Theorem 5.6.2. *Let $f \in Out(V)$. Then $\widetilde{f}$ is linear.*

Proof. Let $A, B, X \in \mathcal{C}l(V)$, and $\alpha, \beta \in \mathbb{R}$. By the definition of the adjoint, and the bilinearity of the scalar product,

$$\begin{aligned} X * \widetilde{f}(\alpha A + \beta B) &= f(X) * (\alpha A + \beta B) \\ &= \alpha(f(X) * A) + \beta(f(X) * B) \\ &= \alpha(X * \widetilde{f}(A)) + \beta(X * \widetilde{f}(B)) \\ &= X * (\alpha \widetilde{f}(A) + \beta \widetilde{f}(B)). \end{aligned} \tag{5.6.2}$$

Since $*$ is non-degenerate, the result follows. □

Theorem 5.6.3. *Let $f \in Out(V)$. Then $\widetilde{f} \in Out(V)$.*

Proof. By linearity we only need to prove the result for k-blades. The $\widetilde{f}$ is linear by Theorem 5.6.2. Let $A_k = a_1 \wedge \cdots \wedge a_k \in \mathcal{C}l(V)$ be a k-blade, and let $X \in \mathcal{C}l(V)$. Then

$$\begin{aligned} X * \widetilde{f}(A_k) &= f(X) * (A_{k-1} \wedge a_k) \\ &= (A_{k-1} \wedge a_k) * f(X) \\ &= (A_{k-1} \wedge a_k) \rfloor f(X) \\ &= A_{k-1} \rfloor (a_k \rfloor f(X)) \end{aligned} \tag{5.6.3}$$

TODO. Since $*$ is non-degenerate, the result follows. □

6 Geometric applications

In this section we derive solutions to problems traditionally approached with various kinds of machinery in linear algebra. We shall see that much of that machinery is subsumed by geometric algebra, providing easily-modifiable formulae with powerful generalizations.

6.1 Cramer's rule

Theorem 6.1.1 (Coordinates of a vector on a basis). *Let $B = \{b_1, \dots, b_k\} \subset V$ be a linearly independent set of invertible vectors, $B_k = b_1 \wedge \dots \wedge b_k$, and $v \in span(B)$. Then*

$$v = \sum_{i=1}^{k} \alpha_i b_i, \tag{6.1.1}$$

where

$$\alpha_i = (b_1 \wedge \dots \wedge b_{i-1} \wedge v \wedge b_{i+1} \wedge \dots \wedge b_k) B_k^{-1}. \tag{6.1.2}$$

Proof. Multiply, with the exterior product, Equation 6.1.1 on both sides from the left by $b_1 \wedge \dots \wedge b_{i-1}$ and from the right by $b_{i+1} \wedge \dots \wedge b_k$. The result follows from the alternation property of the exterior product. □

6.2 Gram-Schmidt orthogonalization

Let $A = \{a_1, \dots, a_k\} \subset V$ be a set of invertible linearly independent vectors. The **Gram-Schmidt orthogonalization** of A is a set $\perp(A) = \{b_1, \dots, b_k\} \subset V$, defined by $b_1 = a_1$, and

$$b_i = (-1)^{i-1}(a_i \cdot a_i)(b_1 \wedge \dots \wedge b_{i-1}) \rfloor (b_1 \wedge \dots \wedge b_{i-1} \wedge a_i)^{-1}, \tag{6.2.1}$$

for $i \in [2, k]$.

Remark 6.2.1. We gave the Gram-Schmidt orthogonalization in a form which uses the previously computed vectors b_i instead of the original vectors a_i. While mathematically equivalent, using the b_i is numerically more stable. This algorithm is often called modified Gram-Schmidt orthogonalization.

Theorem 6.2.2. *If $A = \{a_1, \dots, a_k\} \subset V$ is a set of invertible linearly independent vectors, then $\perp(A) = \{b_1, \dots, b_k\}$ is orthogonal.*

Theorem 6.2.3. *If $A = \{a_1, \dots, a_k\} \subset V$ is an orthogonal set of invertible linearly independent vectors, then $\perp(A) = A$.*

Proof. The result holds by definition for $b_1 = a_1$. Assume $i > 1$. Then

$$\begin{aligned}
b_i &= (-1)^{i-1}(a_i \cdot a_i)(b_1 \wedge \dots \wedge b_{i-1}) \rfloor (b_1 \wedge \dots \wedge b_{i-1} \wedge a_i)^{-1} \\
&= (b_1 \wedge \dots \wedge b_{i-1}) \rfloor (a_i^{-1} \wedge b_1 \wedge \dots \wedge b_{i-1})^{-1} \\
&= (b_1 \cdots b_{i-1})(a_i^{-1} b_1 \cdots b_{i-1})^{-1} \\
&= (b_1 \cdots b_{i-1})(b_1 \cdots b_{i-1})^{-1} a_i \\
&= a_i,
\end{aligned} \tag{6.2.2}$$

for $i \in [2, k]$, where we used Theorems 4.5.9 and 4.10.14. □

6.3 Alternating forms and determinant

Let V and W be vector-spaces over the same field F. A k-**map** from V to W is a function $f_k : V^k \to W$. A k-map is called **multi-linear**, if it is linear in each of its arguments, and **alternating** if it equals zero whenever two of its arguments are equal. A linear combination of two k-maps f_k and g_k is defined by

$$(\alpha f_k + \beta g_k)(a_1, \dots, a_k) = \alpha f_k(a_1, \dots, a_k) + \beta g_k(a_1, \dots, a_k). \tag{6.3.1}$$

A k-**form** on V is a multi-linear k-map from V to F. The set of alternating forms on V is denoted by $\mathrm{Alt}(V)$, and the set of alternating k-forms on V is denoted by $\mathrm{Alt}(V)_k \subset \mathrm{Alt}(V)$. Let $I = \{i_1, \dots, i_k\} \subset \mathbb{N}$ be a set such that $i_1 < \cdots < i_k$, and $\{a_i\}_{i \in I} \subset V$ a set of vectors indexed by I. Then we denote $a_I = (a_{i_1}, \dots, a_{i_k}) \in V^k$. The **exterior product** of alternating forms is a bilinear function $\wedge : \mathrm{Alt}(V)^2 \to \mathrm{Alt}(V)$ such that

$$(f_k \wedge g_l)(a_{[1,k+l]}) = \sum_{(I,J) \in \mathrm{Sh}(k,l)} \mathrm{sgn}(I, J) f_k(a_I) g_k(a_J), \tag{6.3.2}$$

for all $f_k \in \mathrm{Alt}(V)_k$ and $g_l \in \mathrm{Alt}(V)_l$, where the set of (k,l)-**shuffles** is defined by

$$\begin{aligned} \mathrm{Sh}(k,l) = \{(I,J) \in \sigma([1,k+l]) : &I \in \mathbb{N}^k, J \in \mathbb{N}^l, \\ &1 \leq I_1 < \cdots < I_k \leq k+l, \text{ and} \\ &1 \leq J_1 < \cdots < J_l \leq k+l\}. \end{aligned} \tag{6.3.3}$$

Let $\mathrm{Alt}(V)$ be the set of alternating forms on V, and $[\cdot]_k : \mathcal{C}l(V) \to \mathrm{Alt}(V)$, called the k-**form epimorphism**, such that

$$[A]_k(a_1, \dots, a_k) = \widetilde{A} * (a_1 \wedge \cdots \wedge a_k). \tag{6.3.4}$$

Theorem 6.3.1. *Let $A, B \in \mathcal{C}l(V)$. Then*

$$[A]_k \wedge [B]_l = [A \wedge B]_{k+l}. \tag{6.3.5}$$

6.4 Reciprocal bases

Let $B = \{b_1, \dots, b_k\} \subset V$ be linearly independent, and $B_k = b_1 \wedge \cdots \wedge b_k \in \mathcal{C}l(V)$ an invertible k-blade. The **reciprocal** $\{b^1, \dots, b^k\} \subset V$ of B is defined by

$$b^i = (-1)^{i-1}(b_1 \wedge \cdots \wedge \check{b}_i \wedge \cdots \wedge b_k)^{\Vdash B_k},$$

for all $i \in [1,k]$, where the check-mark denotes a missing factor.

Theorem 6.4.1. *Let $B = \{b_1, \dots, b_k\} \subset V$ be linearly independent, $B_k = b_1 \wedge \cdots \wedge b_k \in \mathcal{C}l(V)$ an invertible k-blade, and $\{b^1, \dots, b^k\} \subset V$ the reciprocal of B. Then*

$$b_i \cdot b^j = \delta(i,j), \tag{6.4.1}$$

for all $i, j \in [1,k]$.

Proof.

$$
\begin{aligned}
b_i \cdot b^j &= b_i \cdot (-1)^{j-1}(b_1 \wedge \cdots \wedge \check{b}_j \wedge \cdots \wedge b_k)^{\Vdash B_k} \\
&= (-1)^{j-1} b_i \rfloor \left[(b_1 \wedge \cdots \wedge \check{b}_j \wedge \cdots \wedge b_k) \rfloor B_k^{-1}\right] \\
&= (-1)^{j-1}(b_i \wedge b_1 \wedge \cdots \wedge \check{b}_j \wedge \cdots \wedge b_k) \rfloor B_k^{-1} \\
&= \delta(i,j)(b_1 \wedge \cdots \wedge b_k) \rfloor B_k^{-1} \\
&= \delta(i,j) B_k \rfloor B_k^{-1} \\
&= \delta(i,j),
\end{aligned}
\tag{6.4.2}
$$

where we used Theorem 4.9.3, and Theorem 4.9.4. □

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