Functional Definition of Transformations in Computer Graphics

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1 Introduction

In this paper we give functional definitions for linear, affine, conformal affine, and rigid transformations between vector spaces. We then show that they have the familiar matrix forms when applied from $\mathbb{R}^n$ to $\mathbb{R}^m$. The notation is as follows. Vectors are written in boldface, while scalars are written in normal style. $\mathbf{V}$ and $\mathbf{W}$ are real vector spaces with an inner product. If $\mathbf{x}, \mathbf{y} \in \mathbf{V}$, then $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes their inner product and $\langle \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle$. $|\mathbf{x}| = \sqrt{\langle \mathbf{x} \rangle}$ denotes the norm of $\mathbf{x}$.

2 Transformations

Let $\mathbf{S} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subset \mathbf{V}$, and $U = \{u_1, \cdots, u_n\} \subset \mathbb{R}$. The sum $\sum_{i=1}^n u_i \mathbf{v}_i$ is called a *linear combination*. A linear combination is called an *affine combination* if $\sum_{i=1}^n u_i = 1$. Let $\mathbf{f} : \mathbf{V} \to \mathbf{W}$. The function $\mathbf{f}$ is:

- A *linear transformation* if it preserves linear combinations:

$$\mathbf{f}\left(\sum_{i=1}^n u_i \mathbf{v}_i\right) = \sum_{i=1}^n u_i \mathbf{f}(\mathbf{v}_i).$$

$$\Leftrightarrow$$

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{V} : \forall \alpha, \beta \in \mathbb{R} : \mathbf{f}(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha \mathbf{f}(\mathbf{x}) + \beta \mathbf{f}(\mathbf{y}).$$

- An *affine transformation* if it preserves affine combinations:

$$\mathbf{f}\left(\sum_{i=1}^n u_i \mathbf{v}_i\right) = \sum_{i=1}^n u_i \mathbf{f}(\mathbf{v}_i),$$

  where $\sum_{i=1}^n u_i = 1$.

$$\Leftrightarrow$$

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{V} : \forall t \in \mathbb{R} : \mathbf{f}((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)\mathbf{f}(\mathbf{x}) + t\mathbf{f}(\mathbf{y}).$$

- A *conformal affine transformation* if it scales distances by a constant:

$$\exists s > 0 \in \mathbb{R} : \forall \mathbf{x}, \mathbf{y} \in \mathbf{V} : |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| = s|\mathbf{x} - \mathbf{y}|$$

- A *rigid transformation* if it preserves distances:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{V} : |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

Clearly a rigid transformation is a conformal affine transformation, and a linear transformation is an affine transformation. That a conformal affine transformation is an affine transformation is not straightforward to see and is proven next.

2.1 Conformal affine transformation is an affine transformation

$$\mathbf{f}:\mathbf{V}\to\mathbf{W} \text{ is conformal affine}$$
$$\Rightarrow$$
$$\mathbf{f}:\mathbf{V}\to\mathbf{W} \text{ is affine}$$

Proof. Extending the definitions, the claim is:

$$\exists s>0\in\mathbb{R}:\forall\mathbf{x},\mathbf{y}\in\mathbf{V}:|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|=s|\mathbf{x}-\mathbf{y}|$$
$$\Rightarrow$$
$$\forall\mathbf{x},\mathbf{y}\in\mathbf{V}:\forall t\in\mathbb{R}:\mathbf{f}((1-t)\mathbf{x}+t\mathbf{y})=(1-t)\mathbf{f}(\mathbf{x})+t\mathbf{f}(\mathbf{y}).$$

Let

$$\begin{aligned}\mathbf{z} &= (1-t)\mathbf{x}+t\mathbf{y}\\ \mathbf{a} &= \mathbf{f}(\mathbf{z})-\mathbf{f}(\mathbf{x})\\ \mathbf{b} &= \mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{z})\end{aligned}$$

It then follows that

$$\begin{aligned}\mathbf{f}(\mathbf{x}) &= \mathbf{f}(\mathbf{z})-\mathbf{a},\\ \mathbf{f}(\mathbf{y}) &= \mathbf{b}+\mathbf{f}(\mathbf{z}).\end{aligned}$$

Rewrite the claim in terms of a and b

$$\begin{aligned}\mathbf{f}(\mathbf{z}) &= \mathbf{f}((1-t)\mathbf{x}+t\mathbf{y})\\ &= (1-t)\mathbf{f}(\mathbf{x})+t\mathbf{f}(\mathbf{y})\\ &= (1-t)(\mathbf{f}(\mathbf{z})-\mathbf{a})+t(\mathbf{b}+\mathbf{f}(\mathbf{z}))\\ &= \mathbf{f}(\mathbf{z})-t\mathbf{f}(\mathbf{z})-(1-t)\mathbf{a}+t\mathbf{b}+t\mathbf{f}(\mathbf{z})\\ &= \mathbf{f}(\mathbf{z})-(1-t)\mathbf{a}+t\mathbf{b}\\ &\Leftrightarrow\\ (1-t)\mathbf{a} &= t\mathbf{b}\end{aligned}$$

Cover the singularities

Let us first cover some singularities:

$$\begin{aligned}
\mathbf{b}=0 &\Rightarrow |\mathbf{b}| = |\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{z})| = 0\\
&\Rightarrow s|\mathbf{y}-\mathbf{z}| = 0\\
&\Rightarrow \mathbf{z}=\mathbf{y}\\
&\Rightarrow (1-t)\mathbf{x}+t\mathbf{y}=\mathbf{y}\\
&\Rightarrow t=1 \vee \mathbf{x}=\mathbf{y}\\
&\Rightarrow \mathbf{f}((1-t)\mathbf{x}+t\mathbf{y}) = (1-t)\mathbf{f}(\mathbf{x})+t\mathbf{f}(\mathbf{y})
\end{aligned}$$

$$t=0 \Rightarrow \mathbf{f}((1-t)\mathbf{x}+t\mathbf{y}) = \mathbf{f}(\mathbf{x}) = (1-t)\mathbf{f}(\mathbf{x})+t\mathbf{f}(\mathbf{y})$$

$$t=1 \Rightarrow \mathbf{f}((1-t)\mathbf{x}+t\mathbf{y}) = \mathbf{f}(\mathbf{y}) = (1-t)\mathbf{f}(\mathbf{x})+t\mathbf{f}(\mathbf{y})$$

From now on, assume $t \neq 0$, $t \neq 1$ and $b \neq 0$.

Find a connection between $|\mathbf{a}|$ and $|\mathbf{b}|$

$$\begin{aligned}
|\mathbf{a}| &= |\mathbf{f}(\mathbf{z})-\mathbf{f}(\mathbf{x})|\\
&= s|\mathbf{z}-\mathbf{x}|\\
&= s|(1-t)\mathbf{x}+t\mathbf{y}-\mathbf{x}|\\
&= s|-t\mathbf{x}+t\mathbf{y}|\\
&= |t|s|\mathbf{y}-\mathbf{x}|\\
&= |t||\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})|\\
&= |t||\mathbf{a}+\mathbf{b}|
\end{aligned}$$

$$\begin{aligned}
|\mathbf{b}| &= |\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{z})|\\
&= s|\mathbf{y}-\mathbf{z}|\\
&= s|\mathbf{z}-\mathbf{y}|\\
&= s|(1-t)\mathbf{x}+t\mathbf{y}-\mathbf{y}|\\
&= s|(1-t)\mathbf{x}+(t-1)\mathbf{y}|\\
&= |1-t|s|\mathbf{x}-\mathbf{y}|\\
&= |1-t|s|\mathbf{y}-\mathbf{x}|\\
&= |1-t||\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})|\\
&= |1-t||\mathbf{a}+\mathbf{b}|
\end{aligned}$$

Thus

$$|1-t||\mathbf{a}| = |t||\mathbf{b}|$$

Let

$$c = \left|\frac{t}{1-t}\right|$$

Then

$$|\mathbf{a}| = c|\mathbf{b}|$$

Show that a and b are collinear

Assume $0 < t < 1$.

$$\begin{aligned}
|\mathbf{a}| + |\mathbf{b}| &= (|t| + |1-t|)|\mathbf{a}+\mathbf{b}| = |\mathbf{a}+\mathbf{b}| \\
&\Rightarrow \\
(|\mathbf{a}| + |\mathbf{b}|)^2 &= |\mathbf{a}+\mathbf{b}|^2 \\
&\Rightarrow \\
|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}| &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\langle \mathbf{a}, \mathbf{b} \rangle \\
&\Rightarrow \\
\langle \mathbf{a}, \mathbf{b} \rangle &= |\mathbf{a}||\mathbf{b}| \\
&\Rightarrow \\
\exists k \in \mathbb{R} : \mathbf{a} &= k\mathbf{b}
\end{aligned}$$

Assume $t < 0$ or $t > 1$.

$$\begin{aligned}
||\mathbf{a}| - |\mathbf{b}|| &= |(|t| - |1-t|)|\mathbf{a}+\mathbf{b}|| = |\mathbf{a}+\mathbf{b}| \\
&\Rightarrow \\
(|\mathbf{a}| - |\mathbf{b}|)^2 &= |\mathbf{a}+\mathbf{b}|^2 \\
&\Rightarrow \\
|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\langle \mathbf{a}, \mathbf{b} \rangle \\
&\Rightarrow \\
\langle \mathbf{a}, \mathbf{b} \rangle &= -|\mathbf{a}||\mathbf{b}| \\
&\Rightarrow \\
\exists k \in \mathbb{R} : \mathbf{a} &= k\mathbf{b}
\end{aligned}$$

Thus for all t (excluding 0 and 1): $\exists k \in \mathbb{R} : \mathbf{a} = k\mathbf{b}$.

Show that $|k| = \left|\frac{t}{1-t}\right|$

$$\begin{aligned}
\exists k \in \mathbb{R} : \mathbf{a} &= k\mathbf{b} \\
&\Rightarrow \\
|\mathbf{a}| &= |k\mathbf{b}| = |k||\mathbf{b}| = c|\mathbf{b}| \\
&\Rightarrow \\
|k| &= c \\
&\Rightarrow \\
|k| &= \left|\frac{t}{1-t}\right|
\end{aligned}$$

Show that $k = \frac{t}{1-t}$

Assume $k = -\frac{t}{1-t}$, then

$$k + 1 = -\frac{t}{1-t} + 1 = \frac{(1-t)-t}{1-t} = \frac{1-2t}{1-t}$$

and

$$\begin{aligned}
|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| &= |\mathbf{a} + \mathbf{b}| = |k\mathbf{b} + \mathbf{b}| = |(k+1)\mathbf{b}| \\
&= \left|\frac{1-2t}{1-t}\right| |\mathbf{b}| \\
&= \left|\frac{1-2t}{1-t}\right| |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})| \\
&= s\left|\frac{1-2t}{1-t}\right| |\mathbf{y} - \mathbf{z}| \\
&= s\left|\frac{1-2t}{1-t}\right| |\mathbf{y} - (1-t)\mathbf{x} - t\mathbf{y}| \\
&= s\left|\frac{1-2t}{1-t}\right| |(1-t)\mathbf{x} + (t-1)\mathbf{y}| \\
&= s\left|\frac{1-2t}{1-t}\right| |1-t||\mathbf{y} - \mathbf{x}| \\
&= s|1-2t||\mathbf{y} - \mathbf{x}|
\end{aligned}$$

A contradiction. Thus $k = \frac{t}{1-t}$.

Finish off the proof

$$\begin{aligned}\mathbf{a} &= k\mathbf{b}\\ &\Leftrightarrow\\ \mathbf{a} &= \frac{t}{1-t}\mathbf{b}\\ &\Leftrightarrow\\ (1-t)\mathbf{a} &= t\mathbf{b}\end{aligned}$$

□

2.2 An alternative definition for a rigid transformation

$$\mathbf{f} : \mathbf{V} \to \mathbf{W} \text{ is rigid}$$
$$\Leftrightarrow$$
$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{V} : \langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{y})\rangle = \langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y}\rangle$$

Proof. '$\Leftarrow$' Choose $\mathbf{x} = \mathbf{z}$:

$$\begin{aligned}\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\rangle &= \langle \mathbf{x} - \mathbf{y}\rangle\\ &\Leftrightarrow\\ |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|^2 &= |\mathbf{x} - \mathbf{y}|^2\\ &\Leftrightarrow\\ |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| &= |\mathbf{x} - \mathbf{y}|\end{aligned}$$

'$\Rightarrow$' The following identity holds for any $\mathbf{a}, \mathbf{b} \in \mathbf{V}$:

$$\langle \mathbf{a}, \mathbf{b}\rangle = \frac{1}{4}\langle \mathbf{a} + \mathbf{b}\rangle + \frac{1}{4}\langle \mathbf{a} - \mathbf{b}\rangle.$$

Thus by using this fact along with that $\mathbf{f}$ is rigid and thus also affine by section 2.1:

$$\begin{aligned}
\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{y}) \rangle &= \frac{1}{4}\langle \mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{z}) - 2\mathbf{f}(\mathbf{y}) \rangle + \frac{1}{4}\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{z}) \rangle \\
&= \frac{4}{4}\left\langle \frac{\mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{z})}{2} - \mathbf{f}(\mathbf{y}) \right\rangle + \frac{1}{4}\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{z}) \rangle \\
&= \frac{4}{4}\left\langle \mathbf{f}\left(\frac{\mathbf{x}+\mathbf{z}}{2}\right) - \mathbf{f}(\mathbf{y}) \right\rangle + \frac{1}{4}\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{z}) \rangle \\
&= \frac{4}{4}\left\langle \frac{\mathbf{x}+\mathbf{z}}{2} - \mathbf{y} \right\rangle + \frac{1}{4}\langle \mathbf{x} - \mathbf{z} \rangle \\
&= \frac{1}{4}\langle \mathbf{x} + \mathbf{z} - 2\mathbf{y} \rangle + \frac{1}{4}\langle \mathbf{x} - \mathbf{z} \rangle \\
&= \langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} \rangle
\end{aligned}$$

□

2.3 Affine transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m \text{ is affine}$$

$$\Leftrightarrow$$

$$\exists \mathbf{A} \in \mathbb{R}^{m \times n} : \exists \mathbf{b} \in \mathbb{R}^m : \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

Proof. '$\Leftarrow$':

$$\begin{aligned}
\mathbf{f}((1-t)\mathbf{x} + t\mathbf{y}) &= \mathbf{A}((1-t)\mathbf{x} + t\mathbf{y}) + \mathbf{b} \\
&= (1-t)\mathbf{A}\mathbf{x} + t\mathbf{A}\mathbf{y} + ((1-t) + t)\mathbf{b} \\
&= (1-t)(\mathbf{A}\mathbf{x} + \mathbf{b}) + t(\mathbf{A}\mathbf{y} + \mathbf{b}) \\
&= (1-t)\mathbf{f}(\mathbf{x}) + t\mathbf{f}(\mathbf{y})
\end{aligned}$$

'⇒':

$$\begin{aligned}
\mathbf{f}(\mathbf{x}) &= \mathbf{f}\left(\sum_{i=1}^{n} x_i \mathbf{e}_i + \left(1 - \sum_{i=1}^{n} x_i\right)\mathbf{0}\right) \\
&= \sum_{i=1}^{n} x_i \mathbf{f}(\mathbf{e}_i) + \left(1 - \sum_{i=1}^{n} x_i\right)\mathbf{f}(\mathbf{0}) \\
&= \sum_{i=1}^{n} x_i \mathbf{f}(\mathbf{e}_i) - \sum_{i=1}^{n} x_i \mathbf{f}(\mathbf{0}) + \mathbf{f}(\mathbf{0}) \\
&= \sum_{i=1}^{n} x_i (\mathbf{f}(\mathbf{e}_i) - \mathbf{f}(\mathbf{0})) + \mathbf{f}(\mathbf{0}) \\
&= \mathbf{A}\mathbf{x} + \mathbf{b}
\end{aligned}$$

where $\mathbf{A} = [\mathbf{f}(\mathbf{e}_1) - \mathbf{f}(\mathbf{0}), \cdots, \mathbf{f}(\mathbf{e}_n) - \mathbf{f}(\mathbf{0})]$, and $\mathbf{b} = \mathbf{f}(\mathbf{0})$. □

2.4 Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m \text{ is linear}$$

$$\Leftrightarrow$$

$$\exists \mathbf{A} \in \mathbb{R}^{m \times n} : \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

Proof. A linear function is an affine function. For a linear function $\mathbf{f}(\mathbf{0}) = \mathbf{f}(0 * \mathbf{0}) = 0 * \mathbf{f}(\mathbf{0}) = \mathbf{0}$. Thus the result is a corollary of section 2.3. □

2.5 Conformal affine transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m \text{ is conformal affine}$$

$$\Leftrightarrow$$

$$\exists \mathbf{Q} \in \mathbb{R}^{m \times n} : \mathbf{Q}^T\mathbf{Q} = I : \exists \mathbf{b} \in \mathbb{R}^m : \exists s \in \mathbb{R} : \mathbf{f}(\mathbf{x}) = s\mathbf{Q}\mathbf{x} + \mathbf{b}$$

Proof. '⇒' We proved in section 2.1 that a conformal affine transformation is an affine transformation. Thus we know that

$$\exists \mathbf{A} \in \mathbb{R}^{m \times n}, \exists \mathbf{b} \in \mathbb{R}^m : \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

and by that same section we can actually explicitly give $\mathbf{A}$ and $\mathbf{b}$.

$$|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| = s|\mathbf{x}-\mathbf{y}|$$
$$\Leftrightarrow$$
$$|(\mathbf{A}\mathbf{x}+\mathbf{b})-(\mathbf{A}\mathbf{y}+\mathbf{b})| = s|\mathbf{x}-\mathbf{y}|$$
$$\Leftrightarrow$$
$$|\mathbf{A}(\mathbf{x}-\mathbf{y})| = s|\mathbf{x}-\mathbf{y}|$$
$$\Leftrightarrow$$
$$(\mathbf{x}-\mathbf{y})^T\mathbf{A}^T\mathbf{A}(\mathbf{x}-\mathbf{y}) = s^2(\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y})$$
$$\Leftrightarrow$$
$$(\mathbf{x}-\mathbf{y})^T(\mathbf{A}^T\mathbf{A}-s^2I)(\mathbf{x}-\mathbf{y}) = 0$$
$$\Leftrightarrow$$
$$\mathbf{A}^T\mathbf{A}-s^2\mathbf{I} = 0$$
$$\Leftrightarrow$$
$$\mathbf{A}^T\mathbf{A} = s^2\mathbf{I}$$

Define $\mathbf{Q} = \frac{\mathbf{A}}{s}$. It then follows that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\mathbf{f}(\mathbf{x}) = s\mathbf{Q}\mathbf{x}+\mathbf{b}$.

'$\Leftarrow$'

$$\begin{aligned}
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|^2 &= |(s\mathbf{Q}\mathbf{x}+\mathbf{b})-(s\mathbf{Q}\mathbf{y}+\mathbf{b})|^2 \\
&= |s\mathbf{Q}\mathbf{x}-s\mathbf{Q}\mathbf{y})|^2 \\
&= s^2|\mathbf{Q}\mathbf{x}-\mathbf{Q}\mathbf{y})|^2 \\
&= s^2|\mathbf{Q}(\mathbf{x}-\mathbf{y})|^2 \\
&= s^2(\mathbf{x}-\mathbf{y})^T\mathbf{Q}^T\mathbf{Q}(\mathbf{x}-\mathbf{y}) \\
&= s^2(\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y}) \\
&= s^2|\mathbf{x}-\mathbf{y}|^2 \\
&\Leftrightarrow \\
|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| &= s|\mathbf{x}-\mathbf{y}|
\end{aligned}$$

□

2.6 Rigid transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^m \text{ is rigid}$$
$$\Leftrightarrow$$
$$\exists\mathbf{Q}\in\mathbb{R}^{m\times n}:\mathbf{Q}^T\mathbf{Q}=I:\exists\mathbf{b}\in\mathbb{R}^m:\mathbf{f}(\mathbf{x})=\mathbf{Q}\mathbf{x}+\mathbf{b}$$

Proof. A rigid transformation is a conformal affine transformation with scaling 1. Thus the result is a corollary of section 2.5. □

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