The Fourier transform of a gaussian function

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1 Abstract

In this paper I derive the Fourier transform of a family of functions of the form $f(x) = ae^{-bx^2}$. I thank "Michael", Randy Poe and "porky_pig_jr" from the newsgroup sci.math for giving me the techniques to achieve this. The intent of this particular Fourier transform function is to give information about the frequency space behaviour of a Gaussian filter.

2 Integral of a gaussian function

2.1 Derivation

Let

$$f(x) = ae^{-bx^2} \qquad \text{with} \qquad a > 0, \quad b > 0$$

Note that $f(x)$ is positive everywhere. What is the integral I of $f(x)$ over $\mathbb{R}$ for particular a and b?

$$I = \int_{-\infty}^{\infty} f(x)dx$$

To solve this 1-dimensional integral, we will start by computing its square. By the separability property of the exponential function, it follows that we'll get a 2-dimensional integral over a 2-dimensional gaussian. If we can compute that, the integral is given by the positive square root of this integral.

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} f(y)dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y)dydx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ae^{-bx^2} ae^{-by^2} dydx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^2 e^{-b(x^2+y^2)} dydx \\
&= a^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-b(x^2+y^2)} dydx
\end{aligned}$$

Now we will make a change of variables from (x, y) to polar coordinates (α, r). The determinant of the Jacobian of this transform is r. Therefore:

$$
\begin{aligned}
I^2 &= a^2 \int_0^{2\pi} \int_0^{\infty} r e^{-br^2} dr d\alpha \\
&= a^2 \int_0^{2\pi} \frac{1}{-2b} \int_0^{\infty} -2bre^{-br^2} dr d\alpha \\
&= \frac{a^2}{-2b} \int_0^{2\pi} \Big/_0^{\infty} e^{-br^2} dr d\alpha \\
&= \frac{a^2}{-2b} \int_0^{2\pi} -1 d\alpha \\
&= \frac{-2\pi a^2}{-2b} \\
&= \frac{\pi a^2}{b}
\end{aligned}
$$

Taking the positive square root gives:

$$
I = a\sqrt{\frac{\pi}{b}}
$$

2.2 Example

Requiring $f(x)$ to integrate to 1 over $\mathbb{R}$ gives the equation:

$$
\begin{aligned}
I &= a\sqrt{\frac{\pi}{b}} = 1 \\
&\Leftrightarrow \\
a &= \sqrt{\frac{b}{\pi}}
\end{aligned}
$$

And substitution of:

$$
b = \frac{1}{2\sigma^2}
$$

Gives the Gaussian distribution $g(x)$ with zero mean and σ variance:

$$
g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}
$$

3 The Fourier transform

We will continue to evaluate the bilateral Laplace transform $B(s)$ of $f(x)$ by using the intermediate result derived in the previous section. The Fourier transform is then given by $F(w) = B(iw)$.

$$\begin{aligned} B(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-sx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a e^{-bx^2} e^{-sx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a e^{-(bx^2+sx)} dx \end{aligned}$$

Next we will complete the square in the exponent.

$$b(x+k)^2 = bx^2 + 2bkx + bk^2$$

By comparing factors of x we see that $2bk = s$ and thus $k = \frac{s}{2b}$. Now:

$$b(x + \frac{s}{2b})^2 - \frac{s^2}{4b} = bx^2 + sx$$

$$\begin{aligned} B(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a e^{-(b(x+\frac{s}{2b})^2 - \frac{s^2}{4b})} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{s^2}{4b}} \int_{-\infty}^{\infty} a e^{-b(x+\frac{s}{2b})^2} dx \end{aligned}$$

Next we will make a change of variables by $x(u) = u - \frac{s}{2b}$.
The determinant of the Jacobian of this transformation is 1. Thus:

$$B(s) = \frac{1}{\sqrt{2\pi}} e^{\frac{s^2}{4b}} \int_{-\infty}^{\infty} a e^{-bu^2} du$$

By using the result from the previous section, the integral is solved as:

$$\begin{aligned} B(s) &= \frac{1}{\sqrt{2\pi}} e^{\frac{s^2}{4b}} a \sqrt{\frac{\pi}{b}} \\ &= \frac{a}{\sqrt{2b}} e^{\frac{s^2}{4b}} \end{aligned}$$

The associated Fourier transform is then:

$$
\begin{aligned}
F(w) &= B(iw) \\
&= \frac{a}{\sqrt{2b}} e^{-\frac{w^2}{4b}}
\end{aligned}
$$

Thus the Fourier transform of a Gaussian function is another Gaussian function. Requiring $f(x)$ to integrate to 1 over $\mathbb{R}$ gives:

$$
B_1(s) = \frac{1}{\sqrt{2\pi}} e^{\frac{s^2}{4b}}
$$

$$
\begin{aligned}
F_1(w) &= B_1(iw) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4b}}
\end{aligned}
$$